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***RESEARCH ARTICLE***

**SOME RANDOM FIXED POINT THEOREMS FOR SELF MAPPING IN BANACH SPACE.**

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***Abstract***

In this paper we establish some random fixed point theorems in Banach Space by new rational expression for Self Mappings which satisfy some contractive conditions.

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## Introduction:-

Banach Fixed Point theorem also called contraction mapping principle or contraction mapping theorem [5]. In metric space gives guarantees the existence and uniqueness of fixed point of some self-mappings of metric space providing constructive method stated by Stephan Banach in 1922. In recent years lot of work have been done in non-linear analysis, the study of non-contraction mapping with the existence of fixed point take attention of some authors in non-linear analysis with the details of existence of a fixed point and also the non-expansive mapping.

Random fixed point theorems in abstract space are useful in the study of non-linear random equations for proving the existence and uniqueness of theorems. It's well known that a physical problems the differential and integral equations are generally non-linear, so Banach contraction principle [7] provides a powerful tool for getting the solution of their equation. Many problems of analysis and applied mathematics are used to find the solutions of non-linear functional equations which can be formulated in terms of finding the fixed points of a non-linear mapping.

**Preliminaries:-**

We recall some definitions and properties of normed linear space.

**Definition 2.1** A set X of elements is called a vector space or linear space or Linear Vector Space over the real's if we have a function + on X x X to X and a function dot (·) on R x X to X that satisfy the following conditions

- $x+y=y+x$
- $(x+y)+z=x+(y+x)$
- There exist $\theta \in x$ such that $x+\theta=x$ for all $x \in X$ .
- $\lambda(x+y)=\lambda x+\lambda y$ , $\lambda \in R$, $x, y \in R$.
- $\lambda(\mu x)=(\lambda\mu)x$ , $\lambda, \mu \in R$, $x \in X$ .
- $0.x=\theta$, $1.x=x$.

Here we call '+' addition and '·' scalar multiplication and $\theta$ is unique.

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**Definition 2.2** Let X be a vector space over the real or complex number, A mapping $\|.\|: X \to R^+$ is called a norm provided that the following conditions are satisfied following conditions

1. $\|x\| = 0 \Leftrightarrow x = 0$
2. $\|x + y\| \le \|x\| + \|y\|$
3. $\|\alpha x\| = |\alpha| \|x\|$

If X is a vector space and $\|.\|$ is a norm on X then the pair $(x, \|.\|)$ is called norms vector space. We called X is a metric space if X is a vector space and $\|.\|$ is norm on X and we define metric d by $d(x, y) = \|x - y\|$ for all x, y in X.

If a normed vector space is complete in this metric then it is called a Banach Space.

**Remark 2.1** If we define a metric space $\rho$ by $\rho(x, y) = \|x - y\|$ then a normed vector space becomes a metric space.

**Definition 2.3** (Banach Space) A Banach Space $(x, \|.\|)$ is a normed vector space such that X is complete under the metric included by the norm $\|.\|$

**Definition 2.4** A sequence $\{x_k\}$ in a normed linear space is said to be a Cauchy sequence if $\|x_k - x_l\| \to 0$ as *k, l* tends to infinity. i.e for given $\delta > 0$ there exist an integer N such that $\|x_k - x_l\| < \delta$ for all *k, l* > N.

**Definition 2.5**[9] Let X is a metric space equipped with a distance d and a mapping f from X to X is said to be Lipschitz continuous if there exist $\lambda \ge 0$ such that,

$$d(f(k), f(l)) \le \lambda d(k, l) \text{ for all } k, l \in X .$$

The λ for which the above inequality holds is the Lipschitz constant of f.
If λ= 1 then f is said to be non-expansive and if λ < 1 then f is said to be a contradiction.

**Fixed Point Theorem For Self Mapping In Banach Space:-**

The Banach Fixed point theorem states as follows

**Theorem 3.1**[7] Let (X, $\delta$ ) complete metric space and $f : X \to X$ is a contraction then f has a unique fixed point.

**Theorem 3.2** Let f be mapping of a Banach space X into itself, if f satisfies the following conditions, $f^2 = I$ where I is identity mapping.

$$\|f(k) - f(l)\| \le \alpha \frac{\|k - l\| \|k - f(k)\| + \|k - f(k)\| \|k - f(l)\| + \|k - f(l)\| \|l - f(k)\|}{\|k - l\| + \|l - f(l)\|}$$
$$+\beta\left(\|k - f(k)\| + \|l - f(l)\|\right) + \delta\left(\|k - f(l)\| + \|l - f(k)\|\right) + \eta\|k - l\|$$

Then for every $k, l$ belongs to X, $0 < \alpha, \beta, \delta \,\&\, \eta < 1$ and $5\alpha + 4\beta + 2\delta + \eta$ is less than 2 then f has a fixed point. If $\alpha + 2\delta + \eta < 1$ then *f* has a unique fixed point.

Proof: Suppose that *k* is a fixed point of the Banach Space *X*.

Let $\frac{(f + I)k}{2}$, $m = f(l)$ and $t = 2l - m$ then we have

$$\|m - k\| = \left\|f(l) - f^2(k)\right\| = \|f(l) - f(f(k)\|$$

$$\leq \alpha \frac{\|l-f(k)\|\|l-f(l)\|+\|l-f(l)\|\|l-f^2(k)\|+\|l-f^2(k)\|\|f(k)-f(l)\|}{\|l-f(k)\|+\|f(k)-f^2(k)\|}$$

$$+\beta\left(\|l-f(l)\|+\|f(k)-f^2(k)\|\right)+\delta\left(\|l-f^2(k)\|+\|f(k)-f(l)\|\right)+\eta\|l-f(k)\|$$

$$\|m-k\| \leq \alpha \frac{\|l-f(k)\|\|l-f(l)\|+\|l-f(l)\|\|k-l\|+\|k-l\|\|f(k)-f(l)\|}{\|l-f(k)\|+\|k-f(k)\|}$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left(\|k-l\|+\|f(k)-f(l)\|\right)+\eta\|l-f(k)\|$$

$$\|m-k\| \leq \alpha \frac{\|l-f(l)\|\|k-f(k)\|}{\|k-l\|}+\|f(k)-f(l)\|$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left(\|k-l\|+\|f(k)-f(l)\|\right)+\eta\|l-f(k)\|$$

$$\|m-k\| \leq \alpha \frac{\|l-f(l)\|\|k-f(k)\|+\|k-l\|\|f(k)-f(l)\|}{\|k-l\|}$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left(\|k-l\|+\|f(k)-f(l)\|\right)+\eta\|l-f(k)\|$$

$$\|m-k\| \leq \alpha \frac{\|l-f(l)\|\|k-f(k)\|}{\|k-l\|}+\|f(k)-f(l)\|$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left(\|k-l\|+\|f(k)-f(l)\|\right)+\eta\|l-f(k)\|$$

$$\|m-k\| \leq \alpha\left[\frac{\|l-f(l)\|\|k-f(k)\|}{\|k-\frac{1}{2}(f+I)x\|}+\|f(k)-f(\tfrac{1}{2}(f+I)x)\|\right]$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left[\|k-\tfrac{1}{2}(f+I)x\|+\|f(k)-f(\tfrac{1}{2}(f+I)x)\|\right]+\eta\|\tfrac{1}{2}(f+I)x-f(k)\|$$

$$\|m-k\| \leq \alpha\left[\frac{\|l-f(l)\|\|k-f(k)\|}{\frac{1}{2}\|k-f(x)\|}+\tfrac{1}{2}\|k-f(k)\|\right]$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left[\tfrac{1}{2}\|k-f(k)\|+\tfrac{1}{2}\|k-f(k)\|\right]+\eta\tfrac{1}{2}\|k-f(k)\|$$

$$\|m-k\| \leq \alpha\left[2\|l-f(l)\|+\tfrac{1}{2}\|k-f(k)\|\right]+\beta\tfrac{1}{2}\|k-f(k)\|$$

$$+\beta\left(\|l-f(l)\|+\|k-f(k)\|\right)+\delta\left[\|k-f(k)\|\right]+\eta\tfrac{1}{2}\|k-f(k)\|$$

$$\|m-k\| \leq \left(\tfrac{\alpha}{2}+\beta+\delta+\tfrac{\eta}{2}\right)\|k-f(k)\|+(2\alpha+\beta)\|l-f(l)\|$$

$$\|t-k\| = \|2l-t-k\| = \|\tfrac{1}{2}(f+I)x-m-x\| = \|f(k)-m\| = \|f(k)-f(l)\|$$

$$\|m-k\| \leq \alpha \frac{\|k-l\|\|k-f(k)\|+\|k-f(k)\|\|k-f(l)\|+\|k-f(l)\|\|l-f(k)\|}{\|k-l\|+\|l-f(l)\|}$$

$$+\beta\left(\|k-f(k)\|+\|l-f(l)\|\right)+\delta\left(\|k-f(l)\|+\|l-f(k)\|\right)+\eta\|k-l\|$$

$$\|m-k\| \leq \alpha \frac{\|k-f(k)\|\left(\|k-l\|+\|k-f(k)\|\right)+\|k-f(l)\|\|l-f(k)\|}{\|k-f(l)\|}$$

$$+\beta\left(\|k-f(k)\|+\|l-f(l)\|\right)+\delta\left(\|k-f(l)\|+\|l-f(k)\|\right)+\eta\|k-l\|$$

$$\|m-k\|\leq\alpha\left[\frac{\|k-f(k)\|\|l-f(l)\|}{\|k-f(l)\|}+\|l-f(k)\|\right]$$

$$+\beta\left(\|k-f(k)\|+\|l-f(l)\|\right)+\delta\left(\|k-f(l)\|+\|l-f(k)\|\right)+\eta\|k-l\|$$

$$\|m-k\|\leq\alpha\left[\frac{\|k-f(k)\|\|l-f(l)\|}{\|k-f(\frac{1}{2}(f+I)x\|}+\|\tfrac{1}{2}(f+I)xl-f(k)\|\right]$$

$$+\beta\left(\|k-f(k)\|+\|l-f(l)\|\right)+\delta\left(\|k-f(\tfrac{1}{2}(f+I)x\|+\|\tfrac{1}{2}(f+I)x-f(k)\|\right)+\eta\|k-\tfrac{1}{2}(f+I)x\|\|m-k\|$$

$$\|m-k\|\leq\alpha\left[\frac{\|k-f(k)\|\|l-f(l)\|}{\frac{1}{2}\|k-f(x)\|}+\tfrac{1}{2}\|k-f(k)\|\right]$$

$$+\beta\left(\|k-f(k)\|+\|l-f(l)\|\right)+\delta\left(\tfrac{1}{2}\|k-f(k)\|+\tfrac{1}{2}\|k-f(k)\|\right)+\eta\tfrac{1}{2}\|k-f(k)\|$$

$$\|m-k\|\leq\alpha\left[2\|l-f(l)\|+\tfrac{1}{2}\|k-f(k)\|\right]+\beta\left(\|k-f(k)\|+\|l-f(l)\|\right)+$$

$$+\delta\left(\|k-f(k)\|\right)+\eta\tfrac{1}{2}\|k-f(k)\|$$

$$\|m-k\|\leq\left[\left(\tfrac{\alpha}{2}+\beta+\delta+\tfrac{\eta}{2}\right)\|k-f(k)\|+(2\alpha+\beta)\|l-f(l)\|\right] \quad \text{(I)}$$

Then,

$$\|m-t\|\leq\|(m-k)-(k-t)\|\leq(\|(m-k)\|+\|(k-t)\|$$

$$+\left[\left(\tfrac{\alpha}{2}+\beta+\delta+\tfrac{\eta}{2}\right)\|k-f(k)\|+(2\alpha+\beta)\|l-f(l)\|\right]$$

$$+\left[\left(\tfrac{\alpha}{2}+\beta+\delta+\tfrac{\eta}{2}\right)\|k-f(k)\|+(2\alpha+\beta)\|l-f(l)\|\right]$$

$$\|m-t\|\leq\left[\left(\alpha+2\beta+2\delta+\eta\right)\|k-f(k)\|+(4\alpha+2\beta)\|l-f(l)\|\right]$$

And also,

$$\begin{aligned}\|m-k\|&=\|f(l)-(2l-m)\|\\&=\|f(l)-2l-f(l))\|\\&=2\|l-f(l)\|\end{aligned} \quad \text{(II)}$$

$$2\|l-f(l)\|\leq(\alpha+\chi+2\lambda+2\mu+2\beta+2\delta+\eta)\|k-f(k)\|+(4\alpha+2\lambda+2\beta)\|l-f(l)\|$$

$$\left(2-(4\alpha+2\beta+2\lambda)\right)\|l-f(l)\|$$

$$\leq\left[(\alpha+\chi+2\lambda+2\mu+2\beta+2\delta+\eta)\right]\|k-f(k)\|$$

$$\|l-f(l)\|\leq\psi\|k-f(k)\| \text{ , where } \quad \psi=\frac{\left(\alpha+\chi+2\lambda+2\mu+2\beta+2\delta+\eta\right)}{\left[2-(4\alpha+2\lambda+2\chi\right]}<1$$

Where, $5\alpha+4\beta+2\delta+\eta<2$

Let us consider $\sigma=\frac{1}{2}(f+I)$ then for every $k\in X$ , we have

$$\begin{aligned}\|\sigma^2(k)-\sigma(k)\|&=\|\sigma(l)-l\|\\&=\|\tfrac{1}{2}(f+I)l-l\|\\&=\tfrac{1}{2}\|l-f(l)\|\end{aligned}$$

$$\le \tfrac{\psi}{2}\|k - f(k)\|$$

Then from the definition of $\psi$ we say that $\{\sigma^2(k)\}$ is a Cauchy sequence in *X*. And hence by completeness $\{\sigma^2(k)\}$ converges to some elements $k_0 \in X$ .

Hence ,
$$\lim_{n\to\infty}\sigma^2(k) = k_0$$
$$\therefore\ \sigma(k_0) = k_0$$

i.e. $k_0$ is fixed point of *f* .

About Uniqueness if possible let $l_0 \ne k_0$ is another fixed point of *f* , then

$$\|k_0 - l_0\| = \|f(k_0) - f(l_0)\|$$
$$\le \alpha \frac{\|k_0 - l_0\|\|k_0 - f(k_0)\| + \|k_0 - f(k_0)\|\|k_0 - f(l_0)\| + \|k_0 - f(l_0)\|\|l_0 - f(k_0)\|}{\|k_0 - l_0\| + \|l_0 - f(l_0)\|}$$
$$+\beta\left(\|k_0 - f(k_0)\| + \|l_0 - f(l_0)\|\right) + \delta\left(\|k_0 - f(l_0)\| + \|l_0 - f(k_0)\|\right) + \eta\|k_0 - l_0\|$$
$$\|k_0 - l_0\| \le \alpha\frac{\|k_0 - l_0\|^2}{\|k_0 - l_0\|} + 2\delta\|k_0 - l_0\| + \eta\|k_0 - l_0\|$$
$$\le \alpha\|k_0 - l_0\| + 2\delta\|k_0 - l_0\| + \eta\|k_0 - l_0\|$$
$$\le \alpha\|k_0 - l_0\| + 2\delta\|k_0 - l_0\| + \eta\|k_0 - l_0\|$$
$$\|k_0 - l_0\| = (\alpha + 2\delta + \eta)\|k_0 - l_0\|$$

This is a contradiction.

Hence $k_0 = l_0$ $\therefore$ fixed point is unique.

This completes the proof.

**Theorem 3.3**[8] Let f be mapping of a Banach space X into itself, if f satisfies the following conditions,

$$\|fq - f(fp)\| \le \alpha\frac{\|q - fq\|\|fp - p\|\|q - p\| + \|q - fp\|^3}{\|q - fp\|^2} + \beta\frac{\|fp - p\|\|fp - fq\|\|q - p\| + \|q - fp\|^3}{\|q - fp\|^2}$$
$$+\gamma\left[\|q - fq\| + \|fp - p\|\right] + \delta\left[\|q - p\| + \|fp - fq\|\right] + \eta\left[\|q - fp\|\right]$$
$$\le \alpha\frac{\|q - fq\|\|fp - p\|\tfrac{1}{2}\|p - fp\| + \tfrac{1}{8}\|p - fp\|^3}{\tfrac{1}{4}\|p - fp\|^2} + \beta\frac{\|fp - p\|\left(\|fp - q\| + \|q - fq\|\right)\tfrac{1}{2}\|p - fp\| + \|q - fp\|^3}{\tfrac{1}{4}\|p - fp\|^2}$$
$$+\gamma\left[\|q - fq\| + \|fp - p\|\right] + \delta\left[\tfrac{1}{2}\|p - fp\| + \|fp - q\| + \|q - fq\|\right] + \eta\tfrac{1}{2}\left[\|p - fp\|\right]$$

$f^2$= I, where I is identity mapping

$$\|fp - fq\| \le \alpha\frac{\|q - fq\|\|p - fp\|\|p - fq\| + \|p - q\|^3}{\|p - q\|^2} + \beta\frac{\|q - fq\|\|q - fp\|\|p - fq\| + \|p - q\|^3}{\|p - q\|^2}$$
$$+\gamma\left[\|p - fp\| + \|q - fq\|\right] + \delta\left[\|p - fq\| + \|q - fp\|\right] + \eta\left[\|p - q\|\right]$$

With the equation $10\alpha + 9\beta + 8\gamma + 5\delta + \eta < 4$ and $p \ne q$ , then it has a unique fixed point.

**Theorem 3.3**[6] Let f be mapping of a Banach space X into itself, if f satisfies the following conditions, $f^2$= I, where I is identity mapping then,

$$\| fp - fq)\| \le \alpha \max\left[ \left(\|p-q\|\right), \|p - fp\|, \|q - fq\|, \frac{\|p - fp\|\|q - fq\|}{1+\|p-q\|} \right]$$
$$+\beta\left[\|p - fp\| + \|q - fq\|\right] + \gamma\left[\|p - fq\| + \|q - fp\|\right] + \delta\left[\|p-q\|\right]$$

Then for every $p, q$ belongs to X, $0 < \alpha, \beta, \gamma \,\&\, \delta < 1$ and $4\beta + 3\gamma + 3\alpha + \delta < 2$ is less than 2 then f has a fixed point. If $\alpha + 2\gamma + \delta < 1$ then *f* has a unique fixed point.

## Conclusion:-

In this paper we have presented some random fixed point theorems by new rational expression for Self Mappings in Banach Space which satisfy some contractive conditions


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