The Book of Mathematical Formulas & Strategies

Sohil & Sejal

OmegaLearn.org

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OmegaLearn.org

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Contents

Preface

Motivation

This book was created to share our passion for math with the entire math community. If you have any feedback, find any errors, or think of any important formula that should be added here, please email us at omegalearn.info@gmail.com.

We appreciate your support and will keep working hard to improve the book. We will also raffle out some prizes including Brilliant Premium Subscriptions to anyone who fills out this Raffle Form.

We will update the book regularly and add new formulas and strategies. Please check OmegaLearn.org to get the Latest Version of This Book. The website also provides a comprehensive list of resources for various subjects like Math, Science, Programming, Physics, Chemistry, Biology, and should be useful for students in all grade levels.

Most of the formulas and strategies covered in this book were discussed in our free online classes. All the class videos are available on our YouTube channel. These classes should provide a good foundation for anyone preparing for math competitions.

Information about math competitions

These videos provide some useful strategies for anyone just starting with math competitions or working towards specific competitions like the AMC 10/12.

1. All you need to know about Math Competitions from Elementary to High School
2. How to prepare for AMC 10/12 and qualify for AIME and USA(J)MO

Free AMC 8 Fundamentals Course

AMC 8 Fundamentals Course Details

This course covered the most important topics for the AMC 8 math competition. Here is the list of topics that were covered:

1. Permutations & Combinations
2. Casework, Complementary Counting, Overcounting
3. Probability and Geometric Counting
4. Primes, Factorizations, Number of Factors, Divisibility
5. Modular Arithmetic, Digit Cycles, Algebraic Number Theory
6. Ratios/Proportions/Percents, System of Equations, Speed Time Distance
7. Mean/Median/Mode, Telescoping
8. Pythagorean Theorem/Triples, Area of Complex Shapes
9. Area of Complex Shapes, Extending Lines, Breaking Up Areas
10. Length of Complex Shapes, Angle Chasing

Free AMC 8 Advanced/MATHCOUNTS Course

AMC 8 Advanced/MATHCOUNTS Course Details

This course covered some advanced topics for the AMC 8 and MATHCOUNTS competitions. Here is the list of topics that were covered:

1. Casework, Complementary Counting, Probability, Stars & Bars, Recursion
2. Arithmetic Sequence, Geometric Sequence, System of Equations
3. Number Theory, Money Problems, GCD/LCM, Triangular Numbers
4. Angle Chasing, Similar Triangles, 3D Geometry
5. Omega Learn Math Competition
6. Omega Learn Math Competition Problem Review
7. All you need to know to ACE the AMC 8

Free AMC 10/12 Course

AMC 10/12 Course Details

This course covered the most important topics for the AMC 10/12 math competitions. Here is the list of topics that were covered:

1. Quadratics, Polynomials, Vieta's Formulas, Roots
   Homework Solutions
2. Algebraic Manipulations, Factorizations, SFFT, Sophie Germain's Identity
   Homework solutions
3. Permutations, Combinations, Casework, Complementary Counting
   Homework Solutions
4. Probability, Expected Value, Geometric Probability, States
   Homework Solutions
5. Number Theory: Factors, Multiples, and Bases
   Homework Solutions
6. GCD/LCM, Modular Arithmetic, Diophantine Equations
   Homework Solutions
7. Triangles, Polygons, Basic Trigonometry
   Homework Solutions
8. Circular Geometry, Power of a Point, Cyclic Quadrilaterals
9. Logarithms
10. Meta-solving Techniques - how to find answers without solving the problem

2021 AMC 10/12 Video Solutions

1. 2021 AMC 10A Solutions
2. 2021 AMC 12A Solutions
3. 2021 AMC 10B Solutions
4. 2021 AMC 12B Solutions

Chapter 1

Algebra

1.1 Mean, Median, Mode

Video Link(s)

Mean, Median, Mode

Definition 1.1.1 (Mean/Average).

$$\text{Mean} = \text{average of all terms} = \frac{\text{sum of all terms}}{\text{number of terms}}$$

Definition 1.1.2 (Mode).

$$\text{Mode} = \text{Most common term(s)}$$

Remark 1.1.3
There could be multiple modes. If the problem says "unique mode", it means that there is only one mode.

Definition 1.1.4 (Median). After arranging the numbers in increasing or decreasing order:
If number of terms is odd,

$$\text{Median} = \text{middle number}$$

If number of terms is even,

$$\text{Median} = \text{average of middle two numbers}$$

Definition 1.1.5.

$$\text{Harmonic Mean of Numbers } a_1, a_2, a_3, \ldots, a_n$$

$$= \frac{1}{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}} = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$

This means the harmonic mean is

$$\frac{n}{\text{sum of reciprocals of all } a_k}$$

1.2 Arithmetic Sequences

Video Link(s)

Arithmetic Sequence

Definition 1.2.1 (Arithmetic Sequence)**.** An arithmetic sequence is a sequence of numbers with the same difference between consecutive terms.

$$1, 4, 7, 10, 13, \ldots, 40$$

is an arithmetic sequence because there is always a difference of 3 between consecutive terms.

Remark 1.2.2
Note that an arithmetic sequence can also have a negative common difference. For example, in the arithmetic sequence

$$40, 37, 34, \ldots, 4, 1$$

the common difference is -3.

Definition 1.2.3 (Arithmetic Sequence Notation)**.** In general, the terms of an arithmetic sequence can be represented as:

$$a_1, a_2, a_3, a_4, \ldots, a_n$$

where

- d is the common difference between consecutive terms
- n is the number of terms

Theorem 1.2.4 (nth term in an Arithmetic Sequence)

$$a_n = a_1 + (n-1)d$$

which basically means the nth term of an arithmetic sequence is equal to

$$\text{first term} + (\text{number of terms} - 1)(\text{common difference})$$

We also have that

$$a_n = a_m + (n-m)d$$

which means

$$\text{the nth term} = \text{mth term} + (\text{number of terms} - m)(\text{common difference})$$

Theorem 1.2.5 (Number of Terms in an Arithmetic Sequence)

$$n = \frac{a_n - a_1}{d} + 1$$

Essentially,

$$\text{Number of Terms} = \frac{\text{Last Term} - \text{First Term}}{\text{Common Difference}} + 1$$

Theorem 1.2.6 (Average of Terms in an Arithmetic Sequence)

$$\text{Average of Terms} = \frac{a_1 + a_n}{2}$$

$$\text{Average of Terms} = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

which essentially means

$$\text{Average of Terms} = \frac{\text{First Term} + \text{Last Term}}{2}$$

$$\text{Average of Terms} = \frac{\text{Sum of all Terms}}{\text{Number of Terms}}$$

$$\text{If the number of terms is even, } x = \text{average of middle 2 terms}$$

.

$$\text{If number of terms is odd, } x = \text{middle term}$$

Theorem 1.2.7 (Sum of all Terms in an Arithmetic Sequence)

$$S_n = \frac{a_1 + a_n}{2} \times n$$

which essentially means

$$\text{Sum of All Terms} = \text{Average of Terms } \times \text{Number of Terms}$$

We can also substitute

$$a_n = a_1 + (n-1)d$$

to get

$$S_n = \frac{2a_1 + (n-1) \cdot d}{2} \times n$$

1.3 Geometric Sequences

Video Link(s)

Geometric Sequences

Definition 1.3.1 (Geometric Sequence)**.** A geometric sequence is a sequence of numbers with the same ratio between consecutive terms.

$$1, 2, 4, 8, 16, 32 \ldots, 1024$$

is a geometric sequence because there is always a ratio of 2 between consecutive terms.

Definition 1.3.2 (Geometric Sequence Notation)**.** In general, the terms of a geometric sequence can be represented as:

$$g_1, g_2, g_3, g_4, \ldots, g_n$$

where

- r is the common ratio between consecutive terms
- n is the number of terms

Remark 1.3.3
Note that a geometric sequence can also have a negative common ratio. For example the sequence $1, -2, 4, -8, \ldots, 512, -1024$ has a common ratio of -2.

Theorem 1.3.4 (nth term in a Geometric Sequence)

$$g_n = g_1 \cdot r^{n-1}$$

which basically means

$$\text{the nth term of a geometric sequence} = \text{first term} \times (\text{common ratio})^{\text{number of terms}-1}$$

A general form for calculating the nth term

$$g_n = g_m \cdot r^{(n-m)}$$

which basically means

$$\text{the nth term} = \text{mth term} \times \text{common ratio}^{(n-m)}$$

Theorem 1.3.5 (Number of Terms in a Finite Geometric Sequence)

$$n = \log_r \left(\frac{g_n}{g_1}\right) + 1$$

Essentially,

Number of Terms = 1 more than the number of times we needed to multiply$r$ by g_1 to get g_n

Theorem 1.3.6 (Sum of all Terms in a Finite Geometric Sequence)

$$S_n = g_1 \frac{(1 - r^n)}{1 - r}$$

which essentially means

$$\text{Sum of All Terms} = \text{First Term} \times \frac{1 - \text{common ratio}^n}{1 - \text{common ratio}}$$

Theorem 1.3.7 (Sum of all Terms in an Infinite Geometric Sequence)
For $-1 < r < 1$,

$$S_\infty = \frac{g_1}{1 - r}$$

Remark 1.3.8
The reason the formula only works for $|r| < 1$ is because if $|r| \geq 1$ the sum will diverge or essentially be infinite. We can only find the sum of a converging geometric sequence for which the sum approaches a constant value. Some Examples:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1 - \frac{1}{2}} = 2$$

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

1.4 Special Series

Theorem 1.4.1 (Sum of Numbers Formula)

$$1 + 2 + 3 + \cdots + n = \frac{(n)(n+1)}{2}$$

Theorem 1.4.2 (Sum of Odd Numbers Formula)

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

In simple terms, the

$$\text{Sum of first n odd numbers} = n^2$$

Theorem 1.4.3 (Sum of Even Numbers Formula)

$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

To intuitively think about it, just take 2 common from each term

$$2(1 + 2 + 3 + \cdots + n) = 2\frac{(n)(n+1)}{2} = n(n+1)$$

In simple terms, the

$$\text{Sum of first n even numbers} = 2 \times \text{sum of first n numbers}$$

Theorem 1.4.4 (Sum of Squares Formula)

$$1^2 + 2^2 + \cdots + n^2 = \frac{(n)(n+1)(2n+1)}{6}$$

Theorem 1.4.5 (Sum of Cubes Formula)

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{(n)(n+1)}{2}\right)^2$$

1.5 Telescoping

Video Link(s)

Telescoping

Concept 1.5.1 (Telescoping)
Expand the first few and last few terms, and cancel out any terms you see.

Remark 1.5.2
Generally, whenever you have long expressions that seem to be hard or impossible to compute manually, telescoping is probably at play.

Concept 1.5.3 (Partial Fraction Decomposition)
Partial fraction decomposition is a telescoping technique in where you split terms into multiple terms in order for terms to cancel. In general, to find the partial decomposition of $\frac{1}{ab}$ for any arbitrary variables a and b, we write the equation

$$\frac{x}{a} + \frac{y}{b} = \frac{1}{ab}$$

and then solve for x and y. For example, the partial fraction decomposition of

$$\frac{1}{n(n+1)}$$

(in this case $a = n$ and $b = n + 1$ is

$$\frac{1}{n} - \frac{1}{n+1}$$

Using partial fraction decomposition, we can telescope and evaluate expressions easily.

1.6 Speed, Distance, and Time

Video Link(s)

Speed, Distance, and Time

Theorem 1.6.1

$$\text{Distance} = \text{Speed} \times \text{Time}$$

Equivalently,

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

Theorem 1.6.2

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

Remark 1.6.3
A common mistake is to assume that average speed is the averages of all speeds (especially when the distance you are traveling at each of those speeds are the same). Remember, that's not true unless you are traveling at those speeds for the same amount of time!

1.7 Work, Rate, and Time

Theorem 1.7.1

$$\text{Work} = \text{Rate} \times \text{Time}$$

Equivalently,

$$\text{Rate} = \frac{\text{Work}}{\text{Time}}$$

$$\text{Time} = \frac{\text{Work}}{\text{Rate}}$$

1.8 System of Equations

Video Link(s)

System of Equations

Concept 1.8.1 (System of Equations Word Problems)
This is one of the most common topics on the AMC 10 especially. The trick to solving these types of problems is to just assign variables to the unknowns in the problem and solve them.

Concept 1.8.2 (Solving Systems of Equations)
Once you've found the system of equations in your word problem and/or the problem itself gives you a system of equation, you should consider one or more of the following methods for solving them.

1. Substitution
2. Elimination
3. Diagonal Product Method (Only for 2 variable Equations)
4. Adding/Subtracting Equations
5. Be on the lookout for common factorization tricks (see factorization section)
6. Constructing Polynomials from your equations using Vieta's Formula (see Polynomials Section)
7. Graphing your equation (I recommend you use graph paper)
8. Using Symmetry
9. Creating a geometric setup using geometric formulas such as the
   - Law of Cosines
   - Heron's Formula
   - Sin area formula
   - Stewart's theorem
   - Trig Identities

   (See the geometry section for information about these formulas)

1.9 Polynomials

Video Link(s)

Polynomials

Concept 1.9.1 (Discriminant)
In the quadratic formula,

$$b^2 - 4ac$$

(the part inside the square root) is the discriminant of the quadratic.

1. If the discriminant $b^2 - 4ac$ is 0, then the quadratic has a double or repeated root

2. If the discriminant $b^2 - 4ac$ is positive, the quadratic has 2 different real roots

3. If the discriminant $b^2 - 4ac$ is negative, the quadratic has no real roots

Remark 1.9.2
Also note that the quadratic can only have integer solutions if the discriminant $b^2 - 4ac$ is a perfect square.

Theorem 1.9.3 (Quadratic Formula)
The solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Theorem 1.9.4 (Vieta's Formula For Quadratics)

In a quadratic equation
$$ax^2 + bx + c = 0$$
the sum of its roots is
$$\frac{-b}{a}$$
and the product of its roots is
$$\frac{c}{a}$$
.

Theorem 1.9.5 (Vieta's Formula For Higher Degree Polynomials)

In a polynomial
$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$$
, with roots
$$r_1, r_2, r_3, ...r_n$$
the following hold:
$$r_1 + r_2 + r_3 + ... + r_n \text{ (the sum of all products of 1 term)} = -\frac{a_{n-1}}{a_n}$$

$$r_1 r_2 + r_1 r_3 + .. + r_{n-1} r_n \text{ (the sum of all products of 2 terms)} = \frac{a_{n-2}}{a_n}$$

$$r_1 r_2 r_3 + r_1 r_2 r_4 + ... + r_{n-2} r_{n-1} r_n \text{ (the sum of all products of 3 terms)} = -\frac{a_{n-3}}{a_n}$$

$\vdots$
$$r_1 r_2 r_3 \dots r_n \text{ (the sum of all products of n terms)} = (-1)^n \frac{a_0}{a_n}$$

Note that the negative and positive signs alternate. When summing the products for an odd number of terms, we will have a negative sign and otherwise we will have a positive sign.

Theorem 1.9.6 (Rational Root theorem)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x^1 + a_0$$
where a_n is nonzero and each a_i is integral, all rational roots of the polynomial $\pm\frac{p}{q}$ must have

- p divides a_0
- q divides a_n

Essentially,

- the numerators of all fractional roots divide the constant term of the polynomial
- the denominators of all fractional roots divide the coefficient of the largest degree term.

Corollary 1.9.7 (Integer Root theorem)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x^1 + a_0$$
where a_n is nonzero and each a_i is integral

- All integer roots of the polynomial must divide a_0 or the constant term of the polynomial

Remark 1.9.8
This means for polynomials with a leading coefficient of 1 (monic polynomials), the only rational roots will be integers.

Theorem 1.9.9 (Remainder Theorem)
The remainder when when a polynomial $P(x)$ is divided by $x - r$ is $P(r)$

Corollary 1.9.10 (Factor Theorem)
$x - r$ will divide a polynomial $P(x)$ if $P(r) = 0$

This is a direct consequence of the remainder theorem.

Theorem 1.9.11 (Representation of Polynomial in terms of roots)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$$
it can be expressed in the form
$$a_n(x - r_1)(x - r_2)(x - r_3)\ldots(x - r_n)$$
where $r_1, r_2, r_3, \ldots, r_n$ are the n roots of the polynomial.

Corollary 1.9.12 (Representation of Monic Polynomial in terms of roots)
In a polynomial
$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots a_1 x^1 + a_0$$
(with leading coeffecient 1), it can be expressed in the form
$$(x - r_1)(x - r_2)(x - r_3)\ldots(x - r_n)$$
where $r_1, r_2, r_3, \ldots, r_n$ are the n roots of the polynomial.

Theorem 1.9.13 (Fundamental theorem of Algebra)
A polynomial of degree n (the largest term is to the power of n) has n complex roots including multiplicity (for example, a double root would be counted as 2 roots when including multiplicity)

Theorem 1.9.14 (Conjugate Root Theorem)
If $a + bi$ is a root of a polynomial with real coefficients, then $a - bi$ will be too.

Theorem 1.9.15
If $a + b\sqrt{c}$ is a root of a polynomial with rational coefficients, then $a - b\sqrt{c}$ will be too.

1.9.1 Polynomial Manipulations

Concept 1.9.16 (Reciprocal Roots)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$
with roots $r_1, r_2, r_3, \ldots, r_n$,
$$Q(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n$$
will have roots
$$\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n}$$

Essentially, when flipping the coefficients of a polynomial, it will have roots that are reciprocals of the original roots.

Concept 1.9.17 (Roots That Are More or Less)
In a polynomial
$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$
with roots $r_1, r_2, r_3, \ldots, r_n$,
$$Q(x) = a_n(x-k)^n + a_{n-1}(x-k)^{n-1} + ... + a_1(x-k) + a_0$$
will have roots
$$r_1 + k, r_2 + k, r_3 + k, \ldots, r_n + k$$

Remark 1.9.18
Remember, if the roots are k more, than we subtract k from each of the x terms in our polynomial.

Remark 1.9.19
Polynomial Manipulations are useful when evaluating complex expressions in terms of roots. For example, in order to evaluate
$$\frac{1}{(r-3)^3} + \frac{1}{(s-3)^3} + \frac{1}{(t-3)^3}$$
of a polynomial with roots r, s, t, rather than expanding it out and bashing with Vieta's

Formulas, we can simplify construct a new polynomial with roots

$$\frac{1}{(r-3)^3}, \frac{1}{(s-3)^3}, \frac{1}{(t-3)^3}$$

by the above methods and then simply find the sum of the roots of the polynomial.

Theorem 1.9.20 (Newton Sums)

In a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

with roots

$$r_1, r_2, r_3, \ldots, r_n$$

let $S_1 = r_1 + r_2 + \cdots + r_n$

$S_2 = r_1^2 + r_2^2 + \cdots + r_n^2$

$\vdots$

$S_k = r_1^k + r_2^k + \cdots + r_n^k$

$\vdots$

then the following holds true

$a_n S_1 + a_{n-1} = 0$

$a_n S_2 + a_{n-1} S_1 + 2a_{n-2} = 0$

$a_n S_3 + a_{n-1} S_2 + a_{n-2} S_1 + 3a_{n-3} = 0$

$\vdots$

Note that $a_{\text{negative number}} = 0$ as that might show up in your expansion.

Essentially, what this is saying is

1. Start off with a S_k value and multiply by it by the leftmost polynomial coefficient.
2. Then, multiply S_{n-1} by the polynomial's coefficient right after it.
3. Continue doing so and summing the products until either
   - $k = 0$ in which case instead of multiplying S_0 by last the last a term we multiply k
   - a_{n-i} becomes 0 in which case we simply add the last term and stop
4. Set your final sum of terms to be equal to 0

Theorem 1.9.21

If

$$x + \frac{1}{x} = a$$

then

$$x^2 + \frac{1}{x^2} = a^2 - 2$$

$$x^3 + \frac{1}{x^3} = a^3 - 3a$$

$$x^4 + \frac{1}{x^4} = (a^2 - 2)^2 - 2$$

Definition 1.9.22 (Symmetric Polynomials)**.** A polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$$

is symmetric if

$$a_n = a_0$$

$$a_{n-1} = a_1$$

$$a_{n-2} = a_2$$

$$a_{n-3} = a_3$$

etc.

Basically, opposite coefficients are equal.

Concept 1.9.23 (Solving Symmetric Polynomials of Even Degree)

To solve a symmetric polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + ... + a_1 x + a_0$ of even degree,

- Divide by $x^{\frac{n}{2}}$
- Group the x^k and $\frac{1}{x^k}$ terms together
- Make the substitution
$$y = x + \frac{1}{x}$$
and write all the terms in your expression that way
- Solve the reduced polynomial

Remark 1.9.24
Note that by each Newton Sum Equation, we can iteratively calculate each P_k rather than having to bash with Vieta's Formulas.

1.10 Algebraic Manipulations

Video Link(s)
Algebraic Manipulations

1.10.1 Quadratic Factorizations

Theorem 1.10.1 (Exponent Rules)

$$x^{-a} = \frac{1}{x^a}$$
$$x^a \times x^b = x^{a+b}$$
$$x^a \div x^b = x^{a-b}$$
$$(x^a)^b = x^{ab}$$

Theorem 1.10.2 (Difference of Squares)

$$x^2 - y^2 = (x-y)(x+y)$$

Theorem 1.10.3 (Binomial Square Expansions)

$$(x+y)^2 = x^2 + 2xy + y^2 = (x-y)^2 + 4xy$$

$$(x-y)^2 = x^2 - 2xy + y^2 = (x+y)^2 - 4xy$$

$$(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$$

$$(x+y)^2 - (x-y)^2 = 4xy$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz)$$

Video Link(s)

Quadratic Factorizations

1.10.2 Simon's Favorite Factoring Trick

Theorem 1.10.4 (Simon's Favorite Factoring Trick)

$$xy + kx + jy + jk = (x+j)(y+k)$$

Remark 1.10.5

You can generally apply this factorization when you have xy, x, and y terms. After applying the factorization, you can then find all possible values for each of your terms in your factorization (remember negatives!).

Video Link(s)

Simon's Favorite Factoring Trick

1.10.3 Cubic Factorizations

Video Link(s)

Cubic Factorizations

Theorem 1.10.6 (Difference of Cubes)
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Theorem 1.10.7 (Sum of Cubes)
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Theorem 1.10.8 (Binomial Cube Expansions)

$$(x+y)^3 = x^3 + 3xy(x+y) + y^3$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3xy(x-y) - y^3$$
$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y+z)^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-xz-yz)$$

1.10.4 Higher Power Factorizations

Theorem 1.10.9 (nth power Factorizations)
Sum of odd powers

$$x^{2n+1} + y^{2n+1} = (x+y)(x^{2n} - x^{2n-1}y + x^{2n-2}y^2 - \cdots - xy^{2n-1} + y^{2n})$$

Note: The signs in the second term alternate between positive and negative

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$$

Note: The signs in second term are all positive

1.10.5 Sophie Germain's Identity

Theorem 1.10.10 (Sophie Germain's Identity)

$$x^4 + 4y^4 = (x^2 - 2xy + 2y^2)(x^2 + 2xy + 2y^2)$$

Remark 1.10.11
Be on the lookout for 4th powers to apply Sophie Germain's Identity!

Video Link(s)
Sophie Germain's Identity

Concept 1.10.12 (Algebraic Manipulation Techniques)
Here are some ideas for algebraic manipulation:

1. Group terms that are similar together
2. Be on the lookout for factorizations
3. Take advantage of symmetry, you can use to it construct polynomials
4. Make smart substitutions that can simplify your expression (for example, if the term
$$\sqrt{49 - x^2}$$
appears multiple times in your expression just let
$$y = \sqrt{49 - x^2}$$
to simplify it)

1.11 Floor and Ceiling Functions

Definition 1.11.1 (Floor, Ceiling, and Fractional Part Functions)**.**

$$\lfloor x \rfloor = \text{Greatest integer less than or equal to x}$$
$$\lceil x \rceil = \text{Smallest integer greater than or equal to x}$$
$$\{x\} = \text{Fractional part of x (the value after the decimal point)}$$

Remark 1.11.2
[Common Floor and Ceiling Problems Techniques]

Most floor and ceiling problems can be solved using these techniques.

1. Make the substitution $x = \lfloor x \rfloor + \{x\}$
2. Use the floor or ceiling function to find an inequality
   For example, if you know that $y = \lfloor x \rfloor$, then $y \leq x < y + 1$
3. Graph your equations and look for intersection points (we recommend using graph paper)

1.12 Inequalities

Video Link(s)
Inequalities and Optimizations Basics

Theorem 1.12.1 (Trivial Inequality)
For real x, $x^2 \geq 0$
This means all perfect squares are 0 or more.

Corollary 1.12.2 (Completing the Square)
In a quadratic $Q(x) = ax^2 + bx + c$,

- If $a > 0$, then the minimum value of $Q(x)$ is $c - \frac{b^2}{4a}$ and occurs when $x = -\frac{b}{2a}$
- If $a < 0$, then the maximum value of $Q(x)$ is $c - \frac{b^2}{4a}$ and occurs when $x = -\frac{b}{2a}$

Remark 1.12.3
Simple, yet powerful. This is the core of all inequalities and how more advanced inequalities are derived.

The rest of the inequalities are optional for the AMC 10 but are still good to know.

Theorem 1.12.4 (AM-GM Inequality For 2 variables)
For non-negative reals a and b,

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Basically, this means the average of 2 non-negative numbers (arithmetic mean) is always at least as big as the square root of the product of the 2 numbers (the geometric mean).

Note that equality in this expression occurs when $a = b$.

Corollary 1.12.5
The minimum value of $x + \frac{1}{x}$ is 2 and occurs when $x = 1$

Corollary 1.12.6
- The minimum value of $a + b$ (if ab remains constant) occurs when $a = b$
- The maximum value of ab (if $a + b$ remains constant) occurs when $a = b$

Theorem 1.12.7 (AM-GM Inequality For More Variables)
For non-negative reals $a_1, a_2, \ldots a_n$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_n}$$

Note that equality occurs when $a_1 = a_2 \cdots = a_n$. (essentially all the variables are equal).

Another way to say this is

$$\text{Average of n numbers} = \sqrt[n]{\text{product of all n numbers}}$$

Remark 1.12.8
This means in general,

$$\min(a_1 + a_2 + a_3 + \cdots + a_n) = n \cdot \sqrt[n]{a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_n}$$

$$\max(a_1 \cdot a_2 \cdot a_3 \cdots \cdot a_n) = \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$$

Essentially,

$$\min(\text{sum of all numbers}) = n \cdot \sqrt[n]{\text{product of all numbers}})$$

$$\max(\text{product of all numbers}) = (\text{average of all numbers})^n$$

Remark 1.12.9
Generally, we use AM-GM to maximize products or minimize sums.

Theorem 1.12.10 (Weighted AM-GM Inequality)
For non-negative reals, a_i, c_i,

$$\frac{c_1 \cdot a_1 + c_2 \cdot a_2 + \cdots + c_n \cdot a_n}{c_1 + c_2 + \cdots + c_n} \geq \sqrt[c_1+c_2+\cdots+c_n]{a_1^{c_1} \cdot a_2^{c_2} \cdot a_3^{c_3} \cdot \cdots \cdot a_n^{c_n}}$$

Remark 1.12.11
Weighted AM-GM is very similar to AM-GM. One way to visualize weighted AM-GM is that there are c_k number of terms which are all equal to a_k. So instead of writing $a_k + a_k + \cdots + a_k$ c_k times in our sum we simply write $a_k \cdot c_k$, and instead of writing $a_k \cdot a_k \cdot \ldots$ c_k times in our product we simply write $a_k^{c_k}$.

Remark 1.12.12
We use weighted AM-GM when we are trying to make the sum of all terms a constant by multiplying weights to all (or some) the terms. Remember to divide by the weights you multiplied at the end.

Theorem 1.12.13 (Cauchy Schwarz)

For reals a_i and b_i,

$$(a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n)^2 \leq (a_1^2 + a_2^2 + \ldots a_n^2)(b_1^2 + b_2^2 + \ldots b_n^2)$$

This means

$$(\text{the sum of the products of all } a_k \text{ and } b_k)^2 \leq$$

$$\text{product of the sum of squares of all } a_k \text{ and } b_k$$

Equality holds when the ratio of

$$\frac{a_i}{b_i}$$

for all i is the same.

Remark 1.12.14

If you ever forget which side the $\geq$ sign faces, just try a small example like $a_1 = 1$, $a_2 = 2$, $b_1 = 3$, and $b_2 = 4$.

Remark 1.12.15

You generally want to apply Cauchy Schwarz when you are dealing with sums of squares.

Corollary 1.12.16 (Titu's Lemma)

For reals a_i and b_i,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + b_3 + \cdots + b_n}$$

Alternately,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq \frac{(\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n})^2}{b_1 + b_2 + b_3 + \cdots + b_n}$$

Note that this is a direct consequence of Cauchy Schwartz.

Concept 1.12.17 (Techniques For Optimization Problems)

Steps to find maximum/minimum of expressions

1. Try to find another simple expression for maximization is greater than or equal to the expression you are given OR minimization is less than the expression you are given by using 1 (or possibly even more) of the inequalities
   (a) Trivial Inequality
   (b) AM-GM
   (c) Weighted AM-GM (AM-GM weighted and unweighted are useful for maximizing products and minimizing sums)
   (d) Cauchy Schwartz (Cauchy Schwartz is useful when dealing with sums of squares)
2. Verify that the equality case of your inequality holds true with your problem conditions
3. Simplify your equality case and solve for the answer

Chapter 2

Number Theory

2.1 Primes

Definition 2.1.1 (Primes)**.** Primes are numbers that have exactly two factors: 1 and the number itself. Ex. 2, 3, 5, 7, 11, 13, 17, 19, 23, etc. are all primes

Note: 1 is not a prime and 2 is the only even prime.

Remark 2.1.2
In order to check whether a number n is prime, we need to check all the primes that are less than or equal to

$$\sqrt{n}$$

Concept 2.1.3 (Prime Factorization)
Prime factorization is a way to express each number as a product of primes.

Examples:

The prime factorization of 21 is 3×7

The prime factorization of 60 is $2^2 \times 3 \times 5$

Theorem 2.1.4 (Number of Factors of a Number)

If the prime factorization of the number is expressed as:

$$p_1^{e_1} \times p_2^{e_2} \times \cdots \times p_k^{e_k}$$

then the number of factors of this number is

$$(e_1 + 1)(e_2 + 1) \ldots (e_k + 1)$$

Remark 2.1.5

Basically, in order to find the number of factors of a number:

1. Find the prime factorization of the number
2. Add 1 to all of the exponents
3. Multiply them together

Concept 2.1.6 (Divisibility Rules)

2	Last digit is even
3	Sum of digits is divisible by 3
4	Last 2 digits divisible by 4
5	Last digit is 0 or 5
6	Divisible by 2 and 3
7	Take out factors of 7 until you reach a small number that is either divisible or not divisible by 7
8	Last 3 digits are divisible by 8
9	Sum of digits is divisible by 9
10	Last digit is 0
11	Calculate the sum of odd digits (O) and even digits (E). If $\lvert O - E \rvert$ is divisible by 11, then the number is also divisible by 11
12	Divisible by 3 and 4
15	Divisible by 3 and 5

Video Link(s)

Primes and Prime Factorization

Concept 2.1.7 (Digit Cycles)

To calculate large digit(s) of a number a^b, a strategy that may work is to just look for a pattern by computing the first few values of a^b and then seeing that the pattern will repeat for large values of b.

Video Link(s)

More on Digit Cycles

2.2 Integer Sequences

Video Link(s)

Integer Sequences

2.3 Palindromes

Definition 2.3.1. A palindrome is a number that reads the same forward and backward.

Video Link(s)

Palindromes

2.4 GCD/LCM

Definition 2.4.1 (Greatest Common Divisor)**.** The Greatest Common Divisor (GCD) of two or more integers (which are not all zero) is the largest positive integer that divides each of the integers.
Note: This is also known as GCF (Greatest Common Factor), and the terms GCF and GCD are often used interchangeably.

Definition 2.4.2 (Least Common Multiple)**.** The Least Common Multiple (LCM) of two or more integers (which are not all zero) is the smallest positive integer that is divisible by both the numbers.

Concept 2.4.3

GCD/LCM Greatest common divisor of m and n = $GCD(m, n)$ can be found by taking the lowest prime exponents from the prime factorizations of m and n.

Least common multiple of m and n = $LCM(m, n)$ can be found by taking the highest prime exponents from the prime factorizations of m and n.

Theorem 2.4.4

The product of GCD and LCM of two numbers is equal to the product of the two numbers:

$$GCD(m, n) \cdot LCM(m, n) = m \cdot n$$

Theorem 2.4.5

If two numbers have a common factor c, then

$$\gcd(ac, bc) = c \cdot \gcd(a, b)$$

Video Link(s)

More on GCD/LCM

Theorem 2.4.6 (Euclidean Algorithm)

The Euclidean algorithm states that

$$gcd(x, y) = gcd(x - ky, y)$$

where $x > y$ and k is a positive integer.

Remark 2.4.7

We can apply the Euclidean Algorithm multiple times to easily find the GCD of large numbers since after applying the Euclidean algorithm, we know have 2 smaller numbers which we can apply the Euclidean Algorithm again until we get 2 very small numbers.

For example,

$$\begin{aligned}\gcd(186, 92) &= \gcd((186 - (2 \cdot 92)), 92)\\ &= \gcd(2, 92)\\ &= \gcd(2, (92 - (2 \cdot 46)))\\ &= \gcd(2, 0)\\ &= 2\end{aligned}$$

Video Link(s)

Euclidean Algorithm

Theorem 2.4.8 (Bezout's Identity)
Integer solutions to the equation

$$ax + by = c$$

will only exist if and only if $\gcd(a, b)$ divides c

Video Link(s)

GCD/LCM Basics

GCD/LCM Advanced

2.5 Modular Arithmetic

Video Link(s)

Modular Arithmetic

Definition 2.5.1.

$$n \equiv a \pmod{b}$$

means the number 'n' leaves the same remainder as 'a' when divided by b

Theorem 2.5.2

If $a = x \pmod n$ and $b \equiv y \pmod n$, then

$$ab \equiv xy \pmod n$$

Theorem 2.5.3

If $a \equiv x \pmod n$, then

$$a^m \equiv x^m \pmod n$$

Theorem 2.5.4 (Euler's Totient Function)

If number n has the prime factorization

$$p_1^{e_1} \cdot p_2^{e_2} \cdot p_3^{e_3} \ldots p_n^{e_n}$$

then

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_n}\right)$$

where $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n.

Steps to find totient of a number

1. Find prime factorization
2. For all primes, calculate and multiply

$$1 - \frac{1}{p_i}$$

3. Multiply this product to the number n to get the totient

Theorem 2.5.5 (Euler's Totient Theorem)

$$a^{\phi(n)} \equiv 1 \pmod n$$

if and only if

$$gcd(a, n) = 1$$

Video Link(s)

Euler's Totient Function
Euler's Totient Theorem

Corollary 2.5.6 (Fermat's Little Theorem)

$$a^{p-1} \equiv 1 \pmod{p}$$

if an only if p is a prime and

$$gcd(a, n) = 1$$

Note that this follows directly from Euler's Totient Theorem.

Fact 2.5.7. For any integer x,

$$x^2 \equiv 0, 1 \pmod{3}$$

$$x^2 \equiv 0, 1 \pmod{4}$$

Concept 2.5.8

If a is denoted as the modular inverse of $b \pmod{n}$, then

$$ab \equiv 1 \pmod{n}$$

We also write that

$$a^{-1} \equiv b \pmod{n}$$

since a and b are inverses $\pmod{n}$.

Theorem 2.5.9 (Wilson's Theorem)

$$(p-1)! \equiv p - 1 \equiv -1 \pmod{p}$$

Theorem 2.5.10 (Chinese Remainder Theorem)

If a positive number x satisfies

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\vdots \\ x &\equiv a_k \pmod{n_k} \end{aligned}$$

where all n_i are relatively prime, then x has a unique solution $\pmod{n_1 \cdot n_2 \cdot n_3 \dots n_k}$

Remark 2.5.11

Be careful! This may not necessarily be true if any n_i share common factors as then congruences might contradict each other.

Video Link(s)

Chinese Remainder Theorem

Concept 2.5.12 (Solving Linear Congruences)
To solve a linear congruence with 2 congruences you can either

- Guess and Check until you reach a value that works and satisfies both mods
- Algebraic Method
  1. Find 2 congruences
  $$n \equiv r_1 \pmod{m_1}$$
  $$n \equiv r_2 \pmod{m_2}$$
  such that b and d are relatively prime
  2. Rewrite them algebraically
  $$n = k(m_1) + r_1$$
  $$n = j(m_2) + r_2$$
  3. Set them equal mod the smaller of m_1 and m_2 (in this case, say $m_2 > m_1$)
  $$k(m_1) + r_1 \equiv r_2 \pmod{m_2} \implies m_1 \equiv (r_2 - r_1) \cdot k^{-1} \pmod{m_2}$$
  4. Guess and check to find the value of
  $$k^{-1} \pmod{m_2}$$
  5. Using the value of what b is $\pmod{d}$, rewrite it algebraically.
  6. Substitute it back into the expression
  $$n = k(m_1) + r_1$$
  7. Convert it back to mods to get the final congruence

Concept 2.5.13
The solution to
$$n \equiv r_1 \pmod{m_1}$$
$$n \equiv r_2 \pmod{m_2}$$
is
$$n \equiv r_1 + m_1(r_2 - r_1) \cdot i$$
where $i \equiv m_1^{-1} \pmod{m_2}$

Remark 2.5.14
To solve a general congruence of more than 2 congruences, just solve them 2 at a time until you are left with just 1 congruence.

Video Link(s)
More on Modular Arithmetic

2.6 Diophantine Equations

Video Link(s)
Diophantine Equations

Definition 2.6.1. A Diophantine equation is a polynomial equation such that the only solutions of interest are the integer ones (an integer solution is such that all the variables have integer values).

Concept 2.6.2 (Strategies to Solve Diophantine Equations)
Here are some ideas on how to solve diophantine equations

- Take mods of different numbers. This is generally useful when you
  1. Show there are no solutions to a Diophantine equation
  2. Show that there are only a specific type of solution
- You can try to bound the possible values of different terms. This is generally useful when there are a finite number of solutions to your Diophantine equations
- Factoring, using the various factorizations (see the algebra section on this), can help find all the solutions
- Make substitutions to simplify your Diophantine equation
- Look for conditions on what must be multiples/divisors of your variables and rewrite your Diophantine equation in terms of that

2.7 Bases

Definition 2.7.1 (Bases)**.** A number expressed in base-n is similar to base 10 except instead of regrouping to a new place value every 10, we regroup every n.

A number in base n with digits a_m, $a_{m-1} \dots a_2$, a_1, a_0 can be expressed in the form

$$a_0 \cdot n^0 + a_1 \cdot n^1 + a_2 \cdot n^2 + \dots a_{m-1} \cdot n^{m-1} + a_m \cdot n^m$$

where all a_i are digits of the number

Video Link(s)

Base Conversion Basics

AMC 10/12 problems related to bases

Theorem 2.7.2 (Chicken McNugget Theorem)

The maximum value that cannot be expressed as the sum of non-negative multiples of a and b is $ab - a - b$ if a and b are relatively prime.

For relatively prime positive integers a, b ,there are exactly $\frac{(a-1)(b-1)}{2}$ positive integers which cannot be expressed in the form $ma + nb$ where m and n are positive integers.

Remark 2.7.3

This theorem is useful in finding solutions to problems like "the maximum amount of money that can't be created with 3 cent and 5 cent coins".

2.8 P-adic Evaluation

Definition 2.8.1. [Vp Notation] $v_p(n)$ is defined as the exponent of p in the prime factorization of n.

For example, $v_5(75) = 2$ since 75 has 2 factors of 5. $v_2(27) = 0$ since 27 is odd and has no factors of 2.

Theorem 2.8.2 (Vp Exponentiation Formula)

$$v_p(n^k) = k \cdot v_p(n)$$

This basically means the power of p in a number n^k is k times that of the power of p in n.

Theorem 2.8.3 (Vp Product Formula)

$$v_p(ab) = v_p(a) + v_p(b)$$

This basically means

$$\text{exponent of } p \text{ in } ab = \text{ exponent of } p \text{ in } a + \text{ exponent of } p \text{ in } b$$

Theorem 2.8.4 (Vp Division Formula)

$$v_p(\frac{a}{b}) = v_p(a) - v_p(b)$$

This basically means

$$\text{exponent of } p \text{ in } \frac{a}{b} = \text{ exponent of } p \text{ in } a - \text{ exponent of } p \text{ in } b$$

Theorem 2.8.5 (Vp Sum Formula)

If $v_p(a) \neq v_p(b)$, then

$$v_p(a+b) = min(v_p(a), v_p(b))$$

If $v_p(a) = v_p(b)$, then

$$v_p(a+b) \geq min(v_p(a), v_p(b))$$

Theorem 2.8.6 (Lifting the exponent (LTE) for odd primes)
If

$$a \equiv b \not\equiv 0 \pmod{p}$$

then

$$v_p(a^n - b^n) = v_p(a-b) + v_p(n)$$

.
This basically means

$$\text{exponent of } p \text{ in } a^n - b^n = (\text{ exponent of } p \text{ in } a-b) + (\text{ exponent of } p \text{ in } n)$$

if a and b leave the same remainder when divided by p that is not 0.

Remark 2.8.7
Be very careful of the first condition that is bold! Also remember 2 doesn't work for p in this equation! Sometimes when we can't directly apply LTE to our exponent, we can modify our exponents by rewriting our exponent terms.

Theorem 2.8.8 (Lifting the exponent (LTE) for 2)

$$v_2(a^n - b^n) = v_2(a^2 - b^2) + v_2(n) - 1$$

This basically means

$$\text{exponent of 2 in } a^n - b^n = (\text{ exponent of 2 in } a^2 - b^2) + (\text{ exponent of } p \text{ in } n) - 1$$

Theorem 2.8.9 (Legendre's Theorem)

$$v_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$$

This basically means the

$$\text{number of factors of p in } n! =$$

$$\text{the number of multiples of p} \le n + \text{the number of multiples of } p^2 \le n + \dots$$

Chapter 3

Counting and Probability

3.1 Basic Definitions

3.1.1 Factorials

Definition 3.1.1. A Factorial is the product of all positive integers less than or equal to a given positive integer. In other words $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$.

Video Link(s)

Permutations and Combinations

Theorem 3.1.2 (Factorials in Combinatorics)
The number of ways of arranging n objects in a line is $n!$
The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct is $(n-1)!$
The number of ways of arranging n objects in a circle where rotations of the same arrangement aren't considered distinct and reflections of the same arrangement aren't considered distinct is $\frac{(n-1)!}{2}$

3.1.2 Combinations

Definition 3.1.3. A combination is a possible arrangement in a collection of items where the order of the selection does not matter.

Theorem 3.1.4 (Combinations Formula)
The number of ways to choose k objects out of a total of n objects is

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k!}$$

.

Remark 3.1.5
The second way of evaluating choosing expressions is faster for math contests.

Remark 3.1.6
Notice that

$$\binom{n}{k} = \binom{n}{n-k}$$

This is true because we can see choosing k objects on the left hand side is the same as choosing the k objects that will not be selected on the right hand side.

3.1.3 Permutations

Definition 3.1.7. A permutation is a possible arrangement of objects in a set where the order of objects matters.

Theorem 3.1.8 (Permutations Formula)
The number of ways to order k objects out of n total objects is

$${}^{n}P_{k} = \frac{n!}{(n-k)!}$$

.

Remark 3.1.9
Usually, the words permute, order does matter, etc. imply a permutation while the words choose, select, order does not matter, etc. imply a combination.

3.1.4 Subsets

Theorem 3.1.10
The number of subsets of a set of size n is 2^n.

Remark 3.1.11
We have 2 choices for each element in the set: whether to include or not include the element in our subset. This means that one of our subsets is the empty subset, where we decide to not include all of the elements. If a problem trying to count subsets appears, make sure to check whether we should count the empty subset.

3.2 Combinatoric Strategies

Video Link(s)
Casework, Complementary Counting and Overcounting

3.2.1 Complementary Counting

Complementary counting is the problem solving technique of counting the opposite of what we want and subtracting that from the total number of cases. The keyword "at least" indicates that complementary counting may be helpful.

3.2.2 Overcounting

Overcounting is the process of counting more than what you need and then systematically subtracting the parts which do not belong.

3.2.3 Casework

Many counting or probability problems can be solved by dividing a problem into several cases and calculating arrangements and probabilities for each case before summing them together.

Remark 3.2.1
Casework can be a very useful strategy to solve combinatorics problem, especially when there are no other obvious approach.

Remark 3.2.2
Often times, we have to use casework in conjuction with other techniques like complementary counting.

3.3 Advanced Concepts

3.3.1 Word Rearrangements and Counting

Theorem 3.3.1 (Word Rearrangements)
The number of ways to order a word is

$$\frac{n!}{d_1! \times d_2! \times d_3! \times \dots}$$

where n is the number of letters and $d_1, d_2, d_3, \dots$ are the number of times each of the letters that occur more than 1 time appear in the word.

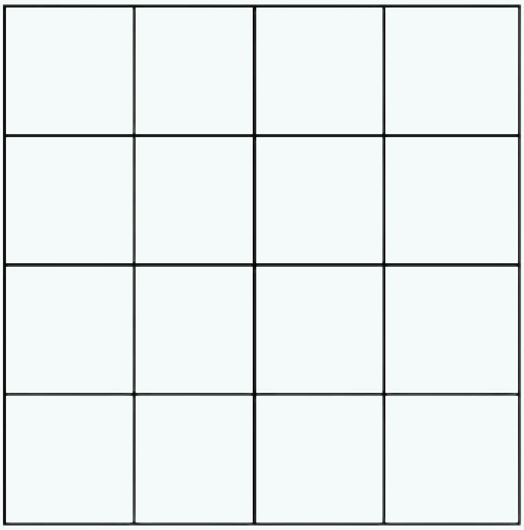
Remark 3.3.2
This is not only true for words! The number of ways of arranging objects or anything else is also the same.

Theorem 3.3.3

The general formula for the number of rectangles of all sizes in a rectangular grid of size $m \times n$ is

$$\binom{m+1}{2} \times \binom{n+1}{2}$$

.



Remark 3.3.4

Each combination of two horizontal lines and two vertical lines creates a unique rectangle. We have

$$\binom{m+1}{2}$$

ways to choose two horizontal lines and

$$\binom{n+1}{2}$$

ways to choose two vertical lines.

Remark 3.3.5

A similar application of combinatorics is the classic problem of how many intersection points occur in a n-gon. Becuase we know an intersection point is created by 2 lines and each line is created by 2 points, the number of intersection points is simply the number of ways to choose 4 points out of n, which is just

$$\binom{n}{4}$$

.

3.3.2 Stars and Bars

Video Link(s)

Stars and Bars Basics

Stars and Bars Advanced

Theorem 3.3.6 (Stars and Bars)

The number of ways to place n objects into k distinguishable bins is

$$\binom{n+k-1}{n}$$

Remark 3.3.7

The reason this is true is because you can consider placing $k-1$ bars in n objects which would have

$$\binom{n+k-1}{n}$$

ways of arranging. It's important you understand this because problems may modify the stars and bars a little...

Remark 3.3.8

Stars and Bars is extremely useful, and can often be adapted based on situations. For example, if each bin has to have at least 1 object in it we assign each bin 1 object to start off with and apply our formula with $n-k$ objects and k distinguishable bins.

3.3.3 Binomial Theorem

Theorem 3.3.9 (Binomial Theorem)
For non-negative n,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n$$

Remark 3.3.10
The binomial theorem has many powerful applications. It's useful for expanding expressions like

$$(x+y)^n$$

Also, if we want to find the value of an expression like

$$(x+y)^n \pmod{n^k}$$

we can just expand the last k terms of the binomial expansion.

Theorem 3.3.11 (Binomial Identity)
The binomial identity states that

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

3.3.4 Combinatorial Identities

Video Link(s)
Combinatorial Identities

Theorem 3.3.12 (Vandermonde's Identity)
Vandermonde's Identity states that

$$\binom{n}{0}\binom{m}{m} + \binom{n}{1}\binom{m}{m-1} + \cdots + \binom{n}{m}\binom{m}{0} = \binom{m+n}{n}$$

.

Theorem 3.3.13 (Special Case of Vandermonde's Identity)

$$\sum_{i=0}^{k} \binom{k}{i}^2 = \binom{2k}{k}$$

Concept 3.3.14 (Pascal's Triangle)

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}$$

Pascal's triangle is pictured below. It can be represented in terms of combinations, which is depicted in the image below.

$$\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}
\end{array}$$

Theorem 3.3.15 (Pascal's Identity)

Pascal's identity states that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Theorem 3.3.16 (Hockey Stick Identity)
Hockey Stick Identity states

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Theorem 3.3.17 (Hockey Stick Identity Generalization)

$$\binom{j}{k} + \binom{j+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1} - \binom{j}{k+1}$$

Theorem 3.3.18 (Choosing Odd Even Identity)
This identity states

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}$$

Remark 3.3.19
These identities can be helpful in combinatorics problem, but their applications may not always be straightforward, so a good approach to many combinatorics problem may be to just manipulate expressions.

3.3.5 Pigeonhole Principle

Theorem 3.3.20 (Generalized Pigeonhole Principle)
If you have at least $nk+1$ objects to distribute into k groups, at least 1 group will have $n+1$ objects.

3.4 Probability and Expected Value

Definition 3.4.1. Probability is the chance something occurs.

Theorem 3.4.2 (Probability)

$$\text{probability} = \frac{\text{Total number of Desired Outcomes}}{\text{Total Outcomes}}$$

Video Link(s)

Probability Basics
Probability Advanced

Definition 3.4.3. Expected value is the weighted average of outcomes.

Theorem 3.4.4 (Expected Value)

The expected value of some event X is

$$\sum x_i \cdot P(x_i)$$

where x_i are the possible values of X and $P(x_i)$ is the probability they occur.

Basically the expected value is just sum the probabilities of events happening times the number or amount of that event

Remark 3.4.5

Often times, in finding the expected value, we can just look for symmetry instead of summing each individual probability times number. For example, to calculate the expected value of a dice roll rather than evaluating

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

we can see that since all rolls from 1 to 6 are equally likely, the expected value is just the average roll which is just the average of the 2 middle terms which is 3.5. (See the arithmetic sequences section)

Video Link(s)

Expected Value

Theorem 3.4.6 (Linearity of Expectation)
For independent or dependent events,

$$E(x_1 + x_2 + \cdots + x_n) = Ex_1 + Ex_2 + \cdots + Ex_n$$

Basically, what this means is that the total expected value of n events is just the sum of the expected values of each individual event.

Remark 3.4.7
This theorem is powerful as it as it allows us to find the expected value of the individual events rather than of the whole thing at once.

3.4.1 Geometric Probability

Video Link(s)
Geometric Probability

Definition 3.4.8. Geometric probability is a way to calculate probability by measuring the number of outcomes geometrically, in terms of length, area, or volume. The key to solving geometric probability is

1. Try a few examples for the different cases, make sure to always mark the extreme cases
2. Try to figure out the region the shape maps out
3. Use geometry to find the area of this region

Remark 3.4.9
Geometric probability can be useful when the number of possible outcomes is infinite.

3.4.2 Principle of Inclusion Exclusion (PIE)

Definition 3.4.10. The principle of inclusion and exclusion (PIE) is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

Definition 3.4.11 (Union Symbol). $|A \cup B|$ is the union of elements in both A and B (duplicates are only written once)

Definition 3.4.12 (Intersection Symbol)**.** $|A \cap B|$ is the intersection of elements in both A and B (only those elements which are in both sets)

Theorem 3.4.13 (Principle of Inclusion Exclusion for 2 Sets)
Given two sets, $|A_1|$ and $|A_2|$

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

.

Basically, we count the number of possibilities in 2 "things" and subtract the duplicates.

Theorem 3.4.14 (Principle of Inclusion Exclusion for 3 Sets)
Given three sets, $|A_1|$, $|A_2|$, $|A_3|$,

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

.

In this formula, we count the number of possibilities in 3 "things", subtract the possibilities that are duplicates in all 3 pairs of sets, and add back the number of duplicates in all 3 sets.

Theorem 3.4.15 (Principle of Inclusion Exclusion Generalized)
If $(A_i)_{1 \leq i \leq n}$ are finite sets, then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots \ + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

.

3.4.3 Bijections

Concept 3.4.16
A bijection is a one to one mapping between the elements of two sets. Bijections can be useful because they allow you to convert difficult problems to ones that can be more easily calculated. When approaching a difficult problem that may involve bijections, you should start out by trying examples and looking for patterns.

Concept 3.4.17 (Recursion)

Recursion is the process solving the problem for small values and writing a recurrence equation to iteratively calculate the values for larger values.

Steps to Solve Recursion Problems:

1. Base Cases: Manually find the values for small values of n
2. Recursion Equation: Look at the different cases for any general value of n (ex. whether the last digit is 0 or 1)
   - If you are stuck, you can try a few small cases and look for a pattern
3. Iteratively calculate higher values of n until you reach your answer

Remark 3.4.18

Note that you can get the answer to many recursion problems by using engineering induction (see the meta-solving section).

Video Link(s)

Recursion

Concept 3.4.19 (States)

We use states in problems when we are trying to find the probability of ”a win” from different positions or turns

When encountering states problems we use the following steps

1. Assign variables to the probabilities of winning from the different positions
2. Write your equations for the probability of winning from each of these positions in terms of the other states
3. Solve your system of equations

Remark 3.4.20 (Symmetry in States)

Always be on the lookout for symmetry in states problems (positions where you have an equal probability of winning from) to help simplify your equations.

Remark 3.4.21
Often times in state problems when you have a lot of states, you may have to write a state recursion equation.

Video Link(s)

Probability States

Chapter 4

Geometry

4.1 Triangles

Video Link(s)

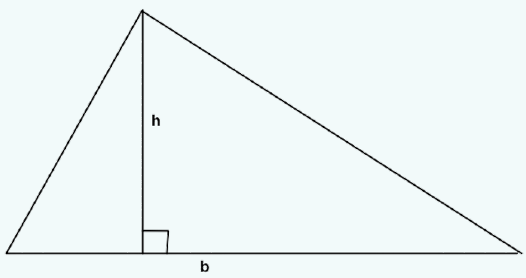
Area of Triangles

4.1.1 Area of a Triangle

There are many ways to calculate the area of a triangle. Here are some of the most useful formulas for calculating the area of a triangle:

Theorem 4.1.1 (Using base and height)
A triangle with base b and height h has an area of

$$\frac{1}{2} \cdot b \cdot h$$



Theorem 4.1.2 (Heron's Formula)

A triangle with sides a, b, c and semiperimeter s has an area of

$$\sqrt{s(s-a)(s-b)(s-c)}$$

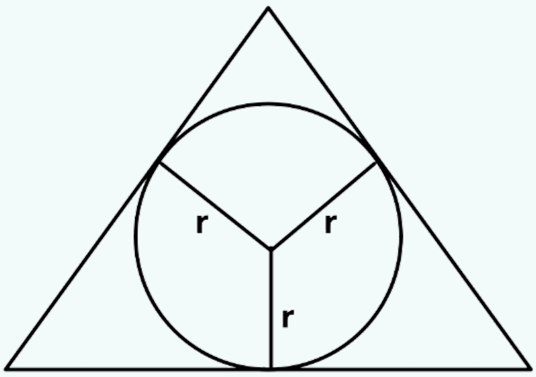
Definition 4.1.3 (In-radius)**.** The inradius of a triangle is the radius of the inscribed circle in the triangle.

Theorem 4.1.4 (Using inradius)

A triangle with inradius r (the radius of the circle that can be inscribed in a triangle) and semiperimeter s has an area of:

$$\text{inradius} \cdot \text{semiperimeter}$$

$$rs$$



Remark 4.1.5

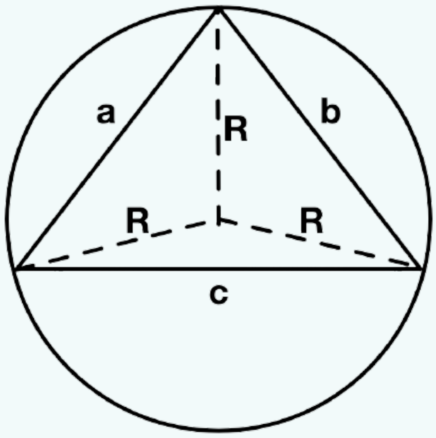
Note that if we know the area of the triangle and it's semi-perimeter, we can apply the inradius formula to find the inradius of the triangle.

Definition 4.1.6 (Circumradius)**.** The circum-radius of a triangle is the radius of circle that a triangle is inscribed in.

Theorem 4.1.7 (Using circumradius)

A triangle with circumradius R (the radius of the circle that the triangle can be inscribed in) and sides a, b, c has an area of

$$\frac{abc}{4R}$$



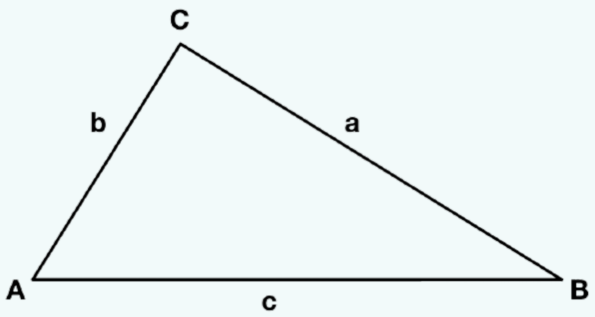
Remark 4.1.8

Similar to the inradius problem, if we know all 3 sides of a triangle, we can apply Heron's and easily calculate the circumradius of the triangle.

Theorem 4.1.9 (Using Trigonometry)

A triangle with 2 sides a and b, which has an angle between the sides to be C, has an area of

$$\frac{1}{2} \cdot a \cdot b \cdot sin(C)$$



Video Link(s)

Using Trigonometry for finding area

Theorem 4.1.10 (Pick's Theorem)

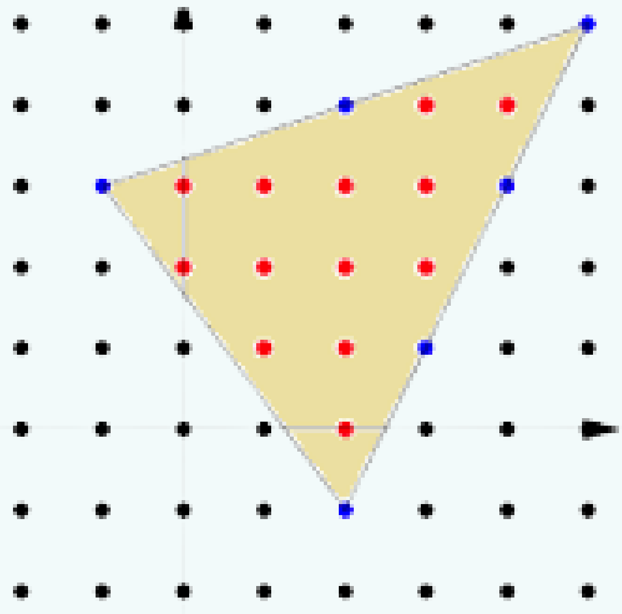
If a polygon has vertices with integer coordinates (lattice points) then the area of the polygon is

$$i + \frac{b}{2} - 1$$

where i is the number of lattice points inside the polygon and b is the number of lattice points on the boundary of the polygon.

Basically, this is

$$\text{Area} = \text{Number of interior lattice points} + \frac{\text{Number of boundary lattice points}}{2} - 1$$



Theorem 4.1.11 (Shoelace Theorem using coordinates)
Suppose the polygon P has vertices (a_1, b_1), (a_2, b_2), ... , (a_n, b_n), listed in clockwise order. Then the area (A) of P is

$$A = \frac{1}{2}\left|(a_1b_2 + a_2b_3 + \cdots + a_nb_1) - (b_1a_2 + b_2a_3 + \cdots + b_na_1)\right|$$

You can also go counterclockwise order, as long as you find the absolute value of the answer.

The Shoelace Theorem gets its name because if one lists the coordinates in a column,

$$\begin{array}{c} (a_1, b_1) \\ (a_2, b_2) \\ \vdots \\ (a_n, b_n) \\ (a_1, b_1) \end{array}$$

and marks the pairs of coordinates to be multiplied,

$$A = \frac{1}{2}\left|\begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{array} \quad - \quad \begin{array}{cc} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{array}\right|$$

Remark 4.1.12 (Intuitive Way of Thinking about Shoelace Theorem)
Steps to Shoelace Theorem

1. Line up all of your polygon's coordinates in a vertical line
2. Repeat your first coordinate at the bottom of your line
3. Let the sum of products of all rightward diagonally pairs be A
4. Let the sum of products of all leftward diagonally pairs be B
5. Find
$$\frac{1}{2}|A - B|$$

4.1.2 Special Triangles

4.1.2.1 Equilateral Triangle

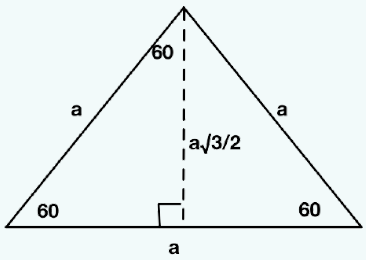
Theorem 4.1.13

If the side length of an equilateral triangle is a

$$\text{Height of the triangle} = \frac{\sqrt{3}}{2}a$$

This follows directly from the $30-60-90$ triangle.

$$\text{Area of the triangle} = \frac{\sqrt{3}}{4}a^2$$



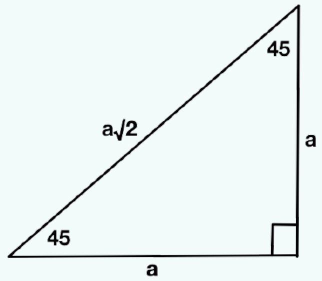
4.1.2.2 45-45-90 Triangle

Theorem 4.1.14

If the side length of a 45-45-90 triangle is a

$$\text{hypotenuse of the triangle} = \sqrt{2} \times \text{side length} = \sqrt{2}a$$

$$\text{Area of the triangle} = \frac{1}{2} \times \text{side length}^2 = \frac{1}{2}a^2$$



4.1.2.3 30-60-90 Triangle

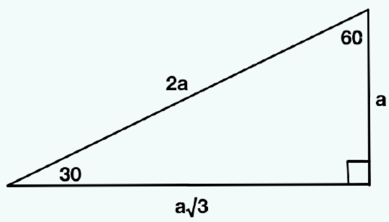
Theorem 4.1.15

If the short leg length of a 30-60-90 triangle is a

$$\text{Long Leg of the triangle} = \sqrt{3} \times \text{short leg} = \sqrt{3}a$$

$$\text{hypotenuse of the triangle} = 2 \times \text{short leg} = 2a$$

$$\text{Area} = \frac{\sqrt{3}}{2} \times \text{short leg}^2 = \frac{\sqrt{3}}{2}a^2$$



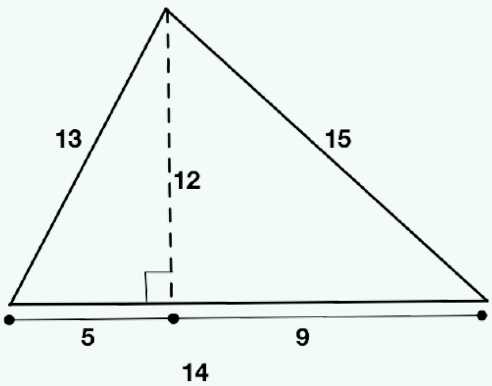
4.1.2.4 13-14-15 Triangle

Theorem 4.1.16

If the three sides of a triangle are 13, 14, and 15, it can be divided into two right triangles with side lengths:

5, 12, 13 and 9, 12, 15

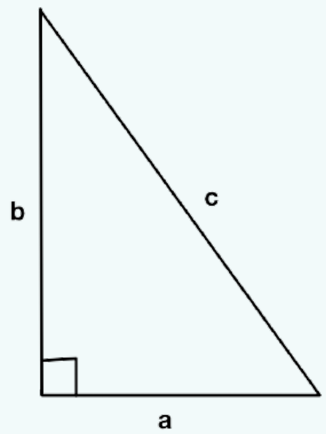
Area of this triangle = 84



4.1.3 Pythagorean Theorem

Theorem 4.1.17 (Pythagorean Theorem)
A right triangle with legs a and b and hypotenuse c satisfies the following relation:

$$c^2 = a^2 + b^2$$



Fact 4.1.18. Important Pythagorean Triples
3, 4, 5
5, 12, 13
7, 24, 25
8, 15, 17
9, 40, 41
20, 21, 29

If all numbers in a pythagorean triple are multiplied by a constant, the resulting numbers still form a pythagorean triple.

For example: These are all pythagorean triples:
3, 4, 5
6, 8, 10
9, 12, 15
12, 16, 20
15, 20, 25

Theorem 4.1.19 (Special Properties of right triangles)

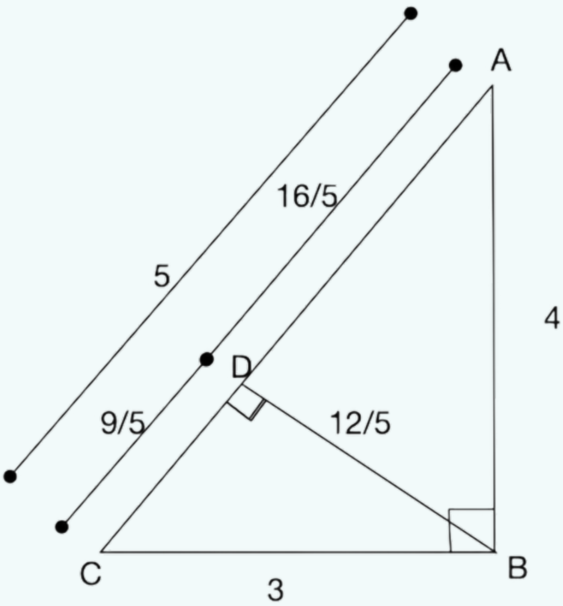
In a right triangle ABC where B is the right angle, the following triangles are similar

$$\triangle ABC \sim \triangle ADB \sim \triangle BDC$$

$$\text{Length of the perpendicular to the hypotenuse (BD)} = \sqrt{\frac{AB \cdot BC}{AC}}$$

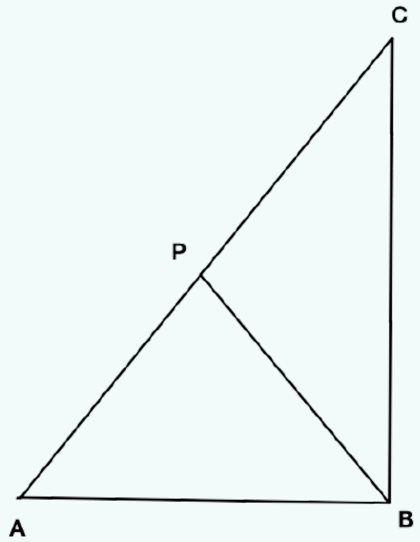
Also note that:

$$\mathrm{AD} \cdot \mathrm{CD} = \mathrm{BD}^2$$
$$\mathrm{AD} \cdot \mathrm{AC} = \mathrm{AB}^2$$
$$\mathrm{CD} \cdot \mathrm{CA} = \mathrm{CB}^2$$



Theorem 4.1.20 (Median in a Right Triangle)

In a right triangle ABC, let the median from point B intersect AC at a point P. Then AP = BP = CP.
Basically, in a right triangle AC is the diameter of the circumcircle, and PC, PA, and PB are radii of the circumcircle.



Video Link(s)

More on Pythagorean Theorem and Area Formulas

4.1.4 Similar Triangles

Concept 4.1.21 (Similarity Test)

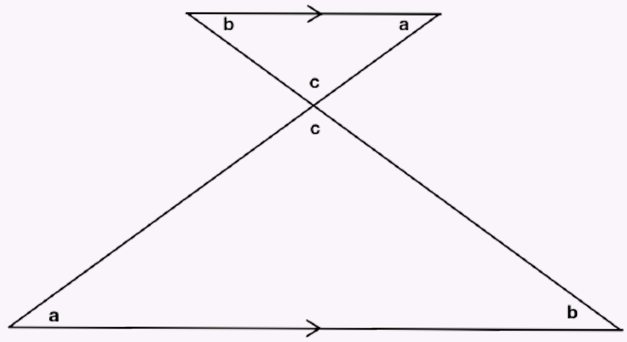
Two triangles are similar if the three angles in the triangle are the same. In other words, the triangles are the same shape multiplied by a scale factor.

In general, triangles are similar if:

- AA similarity: Two angles of the triangles are same, which basically means that the third angle will be equal)
- SAS similarity (Side Angle Side): Two sides are proportional and the angle between the sides is equal
- SSS similarity (Side Side Side): All three sides are proportional
- HL similarity (Hypotenuse Leg): In a right triangle, the hypotenuse and leg are proportional
- LL similarity (LL Leg): In a right triangle, the two legs are proportional

Warning: SSA does not mean triangles are similar

An easy way to detect similar triangles is if bases of triangles are parallel and the sides of the triangles are collinear (see figure below)



Theorem 4.1.22
For similar triangles:

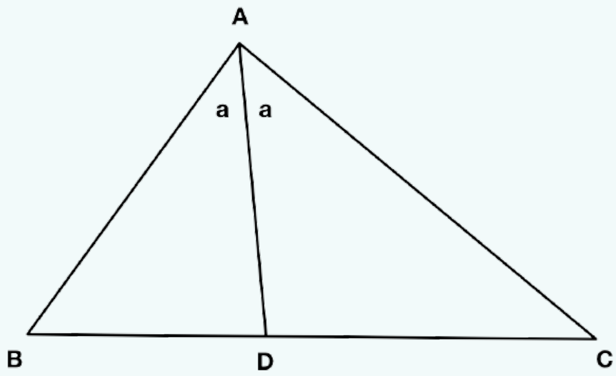
1. All the angles of the triangles are same
2. All corresponding sides have same ratio
3. Area ratio is the square of side length ratio

Video Link(s)
Similar Triangles Basic
Similar Triangles Advanced

4.1.5 Angle Bisector Theorem

Theorem 4.1.23 (Angle Bisector Theorem)
If the line AD bisects angle A, then

$$\frac{AB}{BD} = \frac{AC}{CD}$$



Video Link(s)
More on Angle Bisector Theorem

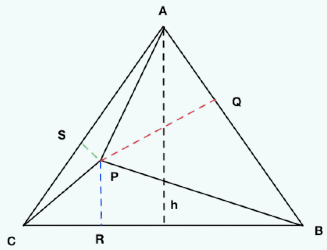
4.1.6 Viviani's Theorem

Theorem 4.1.24

If P is a point inside an equilateral triangle ABC, then the sum of the distances from P to the sides of the triangle is equal to the length of its altitude.

Basically, if the height of the triangle is h, and PQ, PR, and PS are altitudes to AB, BC, and AC respectively, then:

$$PQ + PR + PS = h$$

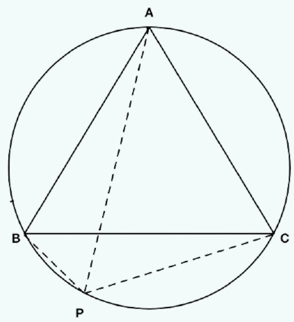


4.1.7 van Schooten's theorem

Theorem 4.1.25 (van Schooten's theorem)

Let P be a point on the minor arc BC of equilateral triangle ABC. Then

$$PA = PB + PC$$



Theorem 4.1.26

For a point P inside an equilateral triangle ABC and side length s,

$$3(PA^4 + PB^4 + PC^4 + s^4) = (PA^2 + PB^2 + PC^2 + s^2)^2$$

4.2 Quadrilaterals

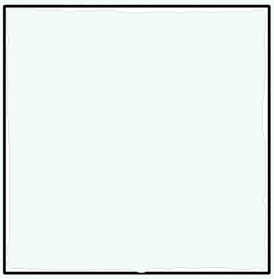
4.2.1 Square

Theorem 4.2.1 (Area of a Square)
Any square with side length s has an area of

$$s^2$$

and a perimeter of

$$4s$$



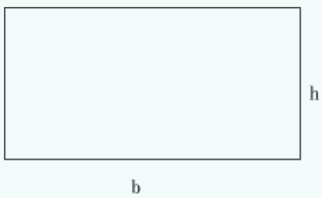
4.2.2 Rectangle

Theorem 4.2.2 (Area of a Rectangle)
Any rectangle with base b and height h has an area of

$$bh$$

and a perimeter of

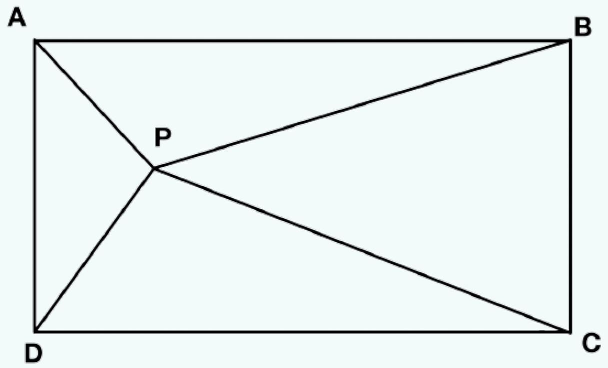
$$2b + 2h$$



Theorem 4.2.3 (British Flag Theorem)

If a point P is chosen inside rectangle ABCD, then

$$(PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$$



4.2.3 Rhombus

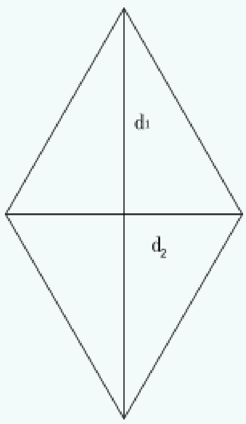
Theorem 4.2.4 (Area of a Rhombus)

A rhombus with diagonals d_1 and d_2 has an area of

$$\frac{1}{2}d_1d_2$$

and a perimeter of

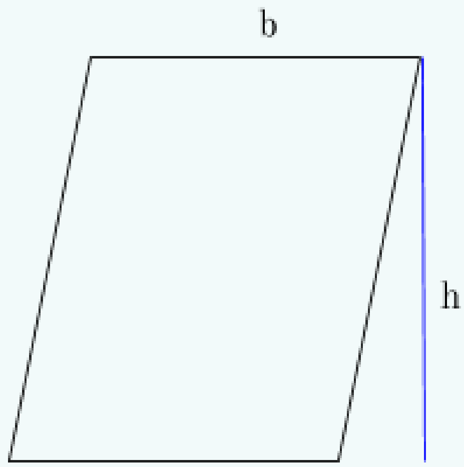
$$2 \times \sqrt{d_1^2 + d_2^2}$$



4.2.4 Parallelogram

Theorem 4.2.5 (Area of a Parallelogram)
A parallelogram with base b and height h has an area of

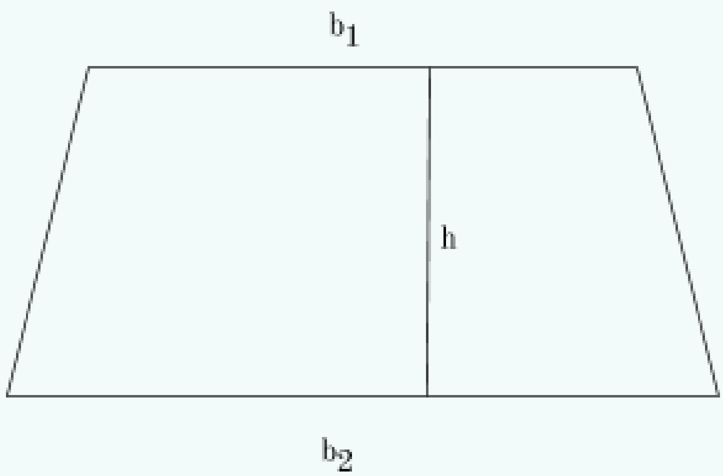
$$bh$$



4.2.5 Trapezoid

Theorem 4.2.6 (Area of a Trapezoid)
A trapezoid with 2 bases b_1 and b_2 and a height h has an area of

$$\frac{b_1 + b_2}{2} \cdot h$$



4.3 Circles

Video Link(s)
Circular Geometry Basics Circular Geometry Advanced, Power of a point, Cyclic Q

4.3.1 Circle Properties

Theorem 4.3.1 (Area and Circumference)
A circle with radius r has

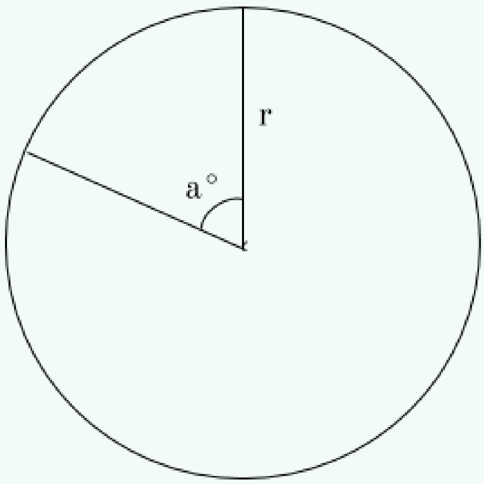
$$\text{Area} = \pi r^2$$

$$\text{Circumference} = 2\pi r$$

Theorem 4.3.2 (Arcs of a circle)
An arc of a circle with radius r and angle a°

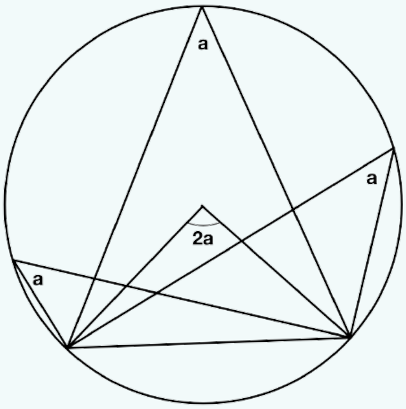
$$\text{Area of a sector} = \pi r^2 \times \frac{a^\circ}{360} = \pi \times \text{radius}^2 \times \text{fraction of circle in sector}$$

$$\text{Length of the arc} = 2\pi r \times \frac{a^\circ}{360} = 2\pi \times \text{radius} \times \text{fraction of circle in sector}$$

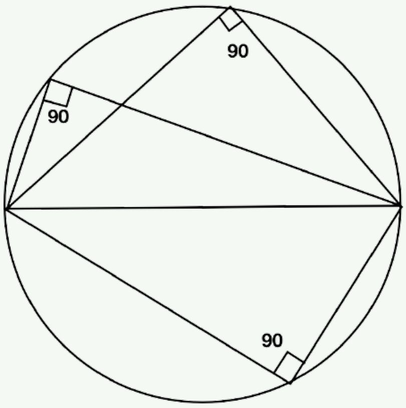


Definition 4.3.3 (Angle of an arc)**.** This is the angle that the arc makes at the center of the circle.

Theorem 4.3.4 (Inscribed Arc Theorem)
The angle formed by an arc in the center or the arc angle is double of the angle formed on the edge.



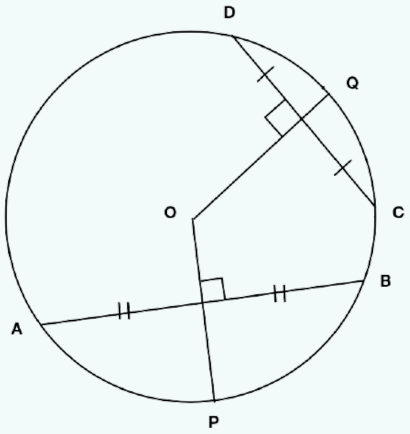
Corollary 4.3.5 (Inscribed Right Triangle)
Inscribed triangle with diameter as one side is always a right triangle.



Definition 4.3.6 (Chord)**.** A Chord is a line segment between any two distinct points on the circle. The diameter of the circle is the longest chord in the circle.

Theorem 4.3.7

The perpendicular bisector of any chord passes through the center. In the figure below, the perpendicular bisectors of AB and CD intersect at the center O.



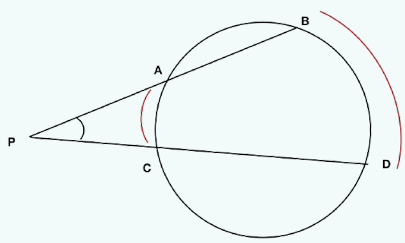
Corollary 4.3.8

- Congruent chords are equidistant from the center of a circle.
- If two chords in a circle are congruent, then their intercepted arcs are congruent.
- If two chords in a circle are congruent, then they determine two central angles that are congruent.

Theorem 4.3.9

The angle marked in the diagram is half of the difference of the 2 red arcs.

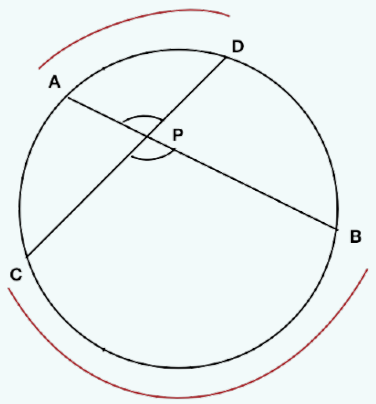
$$\angle APC = \frac{\overset{\frown}{\mathrm{BD}} - \overset{\frown}{\mathrm{AC}}}{2}$$



Theorem 4.3.10

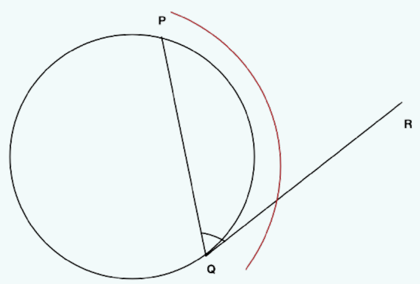
If two chords AB and CD intersect at P, then the $\angle BPC$ and $\angle APD$ are equal to the average of the two arcs.

$$\angle BPC = \angle APD = \frac{\overset{\frown}{\mathrm{BC}} + \overset{\frown}{\mathrm{AD}}}{2}$$



Theorem 4.3.11

If a tangent R intersects the circle at Q, and a chord QP is drawn, then the $\angle RQP$ is equal to half the arc angle



Remark 4.3.12

Circles are really useful for angle chasing so keep an eye out for the inscribed arc theorem that can be used in many angle chasing problems.

Remark 4.3.13

A useful trick to solving angle chasing problems with regular polygons is to draw a circle around the polygon and use the inscribed arc theorem.

Theorem 4.3.14

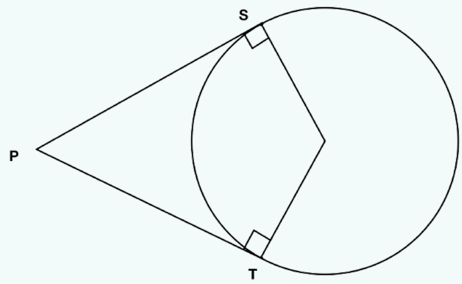
Equal chords mark out equal arcs

This basically means that if you have 2 chords of the same length, the sector of the circle they mark out will be equal

Definition 4.3.15 (Tangent)**.** A tangent is any line from a point external to the circle that just touches the circle.

Theorem 4.3.16 (Right Angle Tangency Point)

If you connect the center of a circle to the point where the circle and a line are tangent, they will form a right angle.



Remark 4.3.17

This property is very useful in circle problems as it allows us to work with right angles. In addition, another helpful technique is drawing useful radii to various points in your diagram as that opens up new information to work with.

4.4 Power of a Point

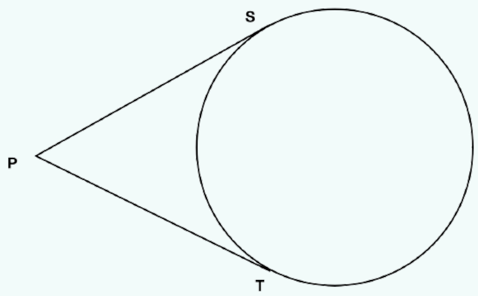
Video Link(s)

Power of a point

Theorem 4.4.1 (Power of Point For 2 Tangents)

From a given point P external to a circle, the two tangents to the circle are equal.

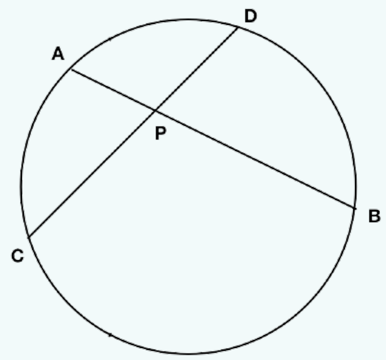
$$PS = PT$$



Theorem 4.4.2 (Power of Point For 2 Secants)

If AB and CD are two secants in a circle, which intersect at a point P, the line segments satisfy the following property:

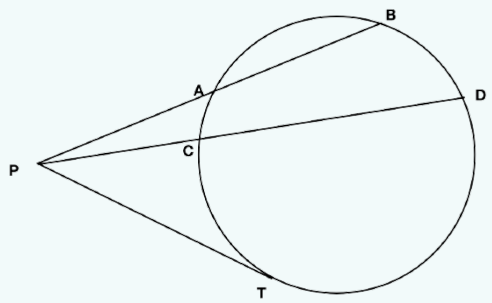
$$PA \cdot PB = PC \cdot PD$$



Theorem 4.4.3 (Power of a Point For Secants and Tangents)

If P is a point external to the circle, PT is a tangent to the circle, and the secants AB and CD intersect P, then the line segments satisfy the following property:

$$PA \cdot PB = PC \cdot PD = PT^2$$



Remark 4.4.4

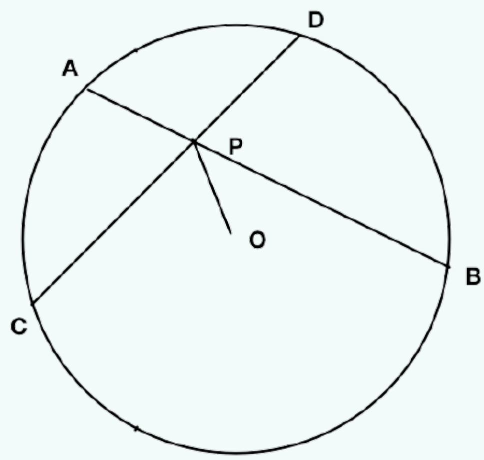
It's a common mistake to get confused and think $PA \cdot AB = PC \cdot CD$. Just remember that all line segments must have the point P (maybe that's why it is called the "Power of a Point")

Theorem 4.4.5

If AB and CD are two secants in a circle, which intersect at a point P inside the circle, the line segments satisfy the following property:

$$PA \cdot PB = PC \cdot PD = r^2 - OP^2$$

where r is the radius of the circle

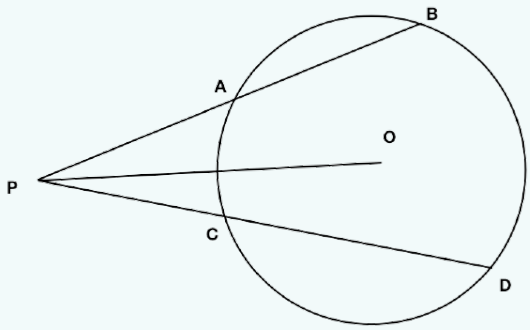


Theorem 4.4.6

If AB and CD are two secants in a circle, which intersect at a point P outside the circle, the line segments satisfy the following property:

$$PA \cdot PB = PC \cdot PD = OP^2 - r^2$$

where r is the radius of the circle



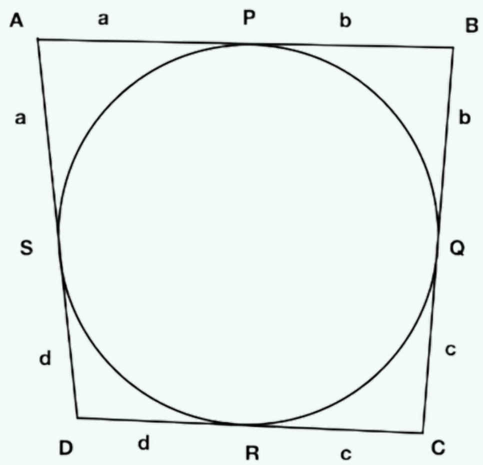
Remark 4.4.7
Power of a point is useful when dealing with circles and chord lengths.

Definition 4.4.8 (Curvature)**.** The curvature of a circle is 1 divided by its radius $\frac{1}{r}$.

Theorem 4.4.9

In a tangential quadrilateral (i.e. one in which a circle can be inscribed) the two sums of lengths of opposite sides are the same.

$$AB + CD = AD + BC$$



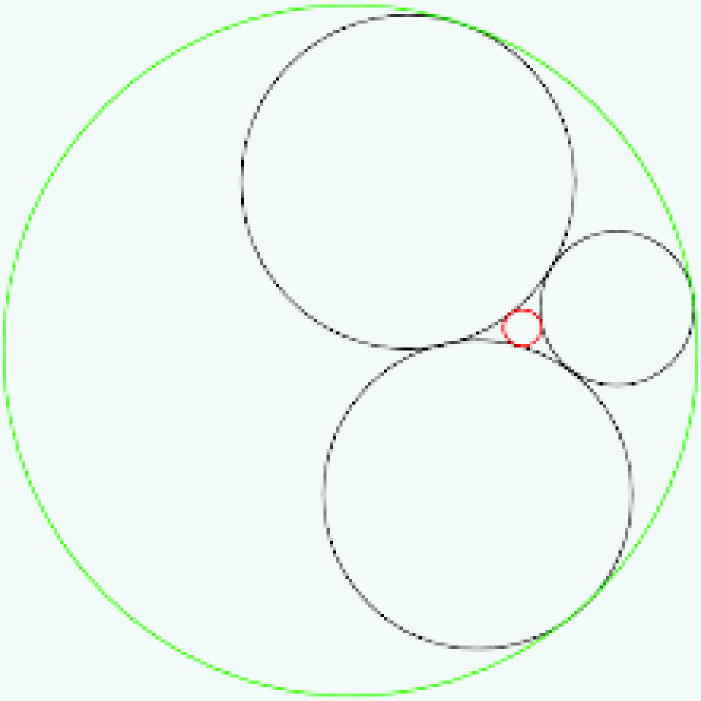
Theorem 4.4.10 (Descartes Theorem)
Let the radii of the 3 black externally tangent circles be $r_1, r_2, \text{and} r_3$. Let the radius of the green circle be s_1. Then we have

$$\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{s_1}\right)^2 = 2\left(\frac{1}{r_1}^2 + \frac{1}{r_2}^2 + \frac{1}{r_3}^2 + \frac{1}{s_1}^2\right)$$

Note the term $\frac{1}{s_1}$ is negative because the green circle is internally tangent to the other circles, which means its curvature is negative.
Also, let the radius of the red circle be s_2. Then we know

$$\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{s_2}\right)^2 = 2\left(\frac{1}{r_1}^2 + \frac{1}{r_2}^2 + \frac{1}{r_3}^2 + \frac{1}{s_2}^2\right)$$

Note the term $\frac{1}{s_2}$ is positive because the red circle is externally tangent to the other circles which means its curvature is positive.

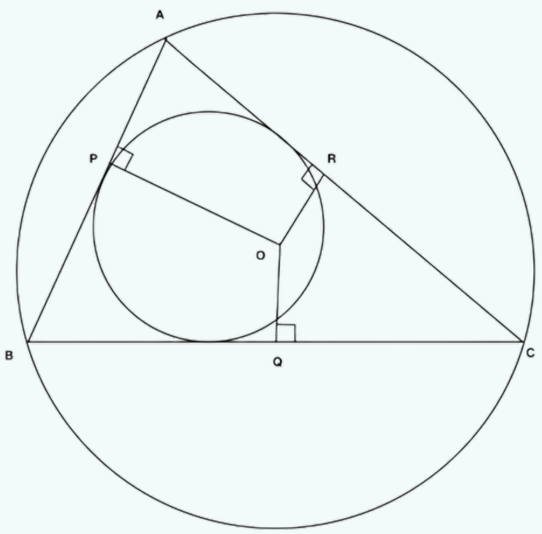


Remark 4.4.11
This formula also works if instead of a circle, one of the circles was a line. In that case, we can just say $\frac{1}{r} = 0$.

Theorem 4.4.12 (Carnot's Theorem)

in a triangle ABC with circumradius R and inradius r, the signed sum of perpendicular distances from the circumcenter O to the sides is:

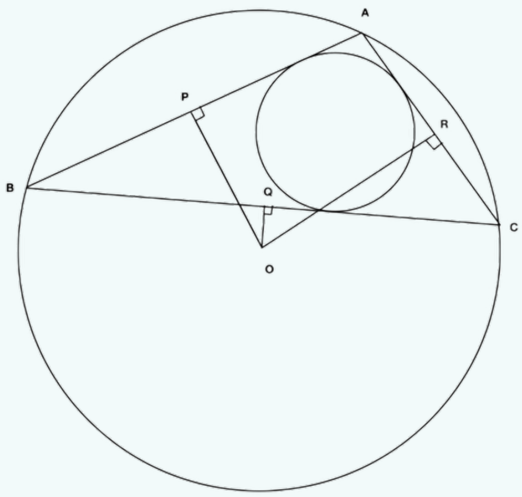
$$OP + OQ + OR = R + r$$



Note: The sign of the distance is chosen to be negative if the line lies outside the triangle as shown below

In the figure below, OQ is negative as the segment lies outside the triangle

$$OP + OR - OP = R + r$$



4.5 Area of Complex Shapes

Concept 4.5.1

Tricks to finding the area of complex shapes

- Divide the shape into "nicer" areas which are easier to calculate
- Extend Lines
  - You generally want to extend lines when they form nicer shapes/areas to work with, such as triangles
- Break up areas
  - A common way to do so is to drop altitudes as doing so generally allows you to form right triangles

Remark 4.5.2

A common technique is to find the area of shapes and then find the area of a shape in terms of a variable (like altitude, inradius, circumradius, etc.) and then solve for that variable.

4.6 Length of complex shapes

Video Link(s)

Length of Complex Shapes

Concept 4.6.1

Finding Length of Complex Shapes

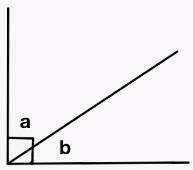
- Having equal angles means equal lengths and vice versa
- Be on the lookout for 90 degree angles, as you can use Pythagorean theorem
- Split the length into multiple components by using some of these techniques
  - Drawing extra lines
  - Dropping Altitudes
- Extending lines to create similar triangles, special triangles, etc. and then subtracting the extra length

4.7 Angle Chasing

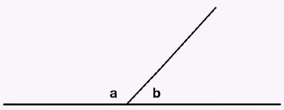
Concept 4.7.1 (Angle Chasing Tricks)

- Sum of Angles in Triangle is 180
- A triangle with 2 angles equal will have their corresponding sides equal and a triangle with 2 sides equal will have their corresponding angles equal (isosceles triangle)
- Opposite angles in intersecting lines are equal
- Corresponding angles in parallel lines are equal
- The angle made by the arc at the center of the circle is double the angle made by the arc at the boundary of the circle

Concept 4.7.2 (Complementary Angle)
Complementary angles are a pair of angles with the sum of 90 degrees

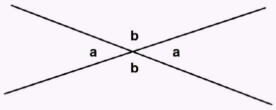


Concept 4.7.3 (Supplementary Angle)
Supplementary angles are a pair of angles with the sum of 180 degrees



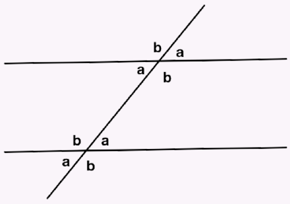
Concept 4.7.4 (Intersecting lines)

When two lines intersect, the vertical angles are equal. Vertical angles are each of the pairs of opposite angles made by two intersecting lines. "Vertical" in this case means they share the same Vertex (corner point), not the usual meaning of up-down.



Concept 4.7.5

Parallel Lines: Corresponding angles equal



4.8 Polygons

Video Link(s)

Area of complex shapes 1 Area of Complex Shapes 2

4.8.1 Angles of a Polygon

Theorem 4.8.1

$$\text{Sum of interior angle of a polygon} = (n-2)\cdot 180$$

$$\text{Interior angle of a regular polygon} = \frac{(n-2)}{n}\cdot 180$$

$$\text{Exterior angle of a regular polygon} = \frac{360}{n}$$

Fact 4.8.2. Important Interior Angles

Number of sides in regular polygon	Interior Angle of regular polygon
3	60
4	90
5	108
6	120
8	135
9	140
10	144

4.8.1.1 Hexagon

Theorem 4.8.3

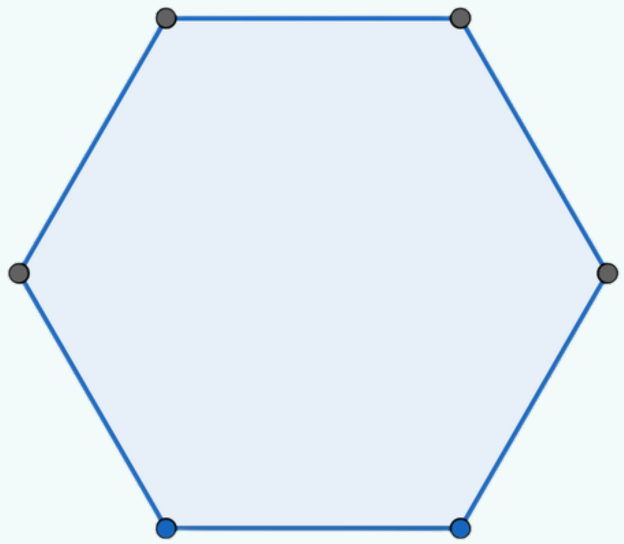
$$\text{Sum of interior angle of a regular hexagon} = (6-2)\cdot 180 = 720$$

$$\text{Interior angle of a regular hexagon} = \frac{(6-2)}{6}\cdot 180 = 120$$

$$\text{Exterior angle of a regular hexagon} = \frac{360}{6} = 60$$

$$\text{Area of a regular hexagon} = 6\cdot\frac{\sqrt{3}}{4}s^2$$

$$\text{Length of the diagonal of a regular hexagon} = 2s$$



4.8.1.2 Octagon

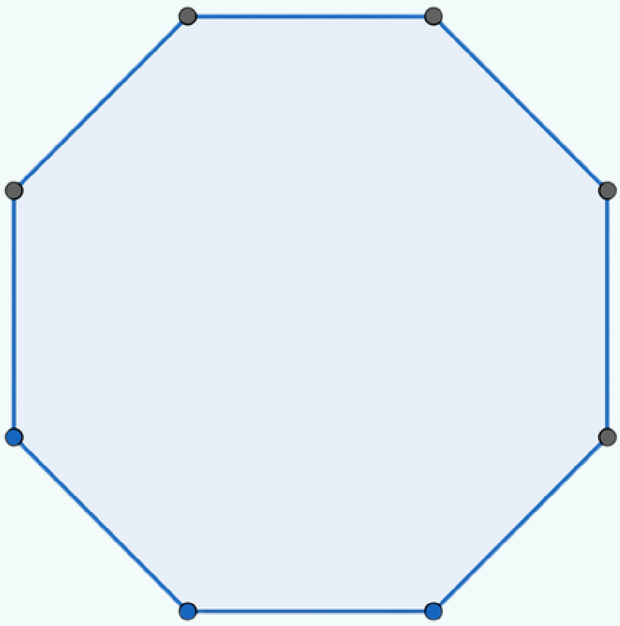
Theorem 4.8.4

$$\text{Sum of interior angle of a regular octagon} = (8-2)\cdot 180 = 1080$$

$$\text{Interior angle of a regular octagon} = \frac{(8-2)}{6}\cdot 180 = 135$$

$$\text{Exterior angle of a regular octagon} = \frac{360}{8} = 45$$

$$\text{Area of a regular octagon} = 2(1+\sqrt{2})s^2$$



Remark 4.8.5
A regular hexagon can be divided into 6 congruent equilateral triangles.

Video Link(s)
Angle Chasing Basics
Angle Chasing Advanced

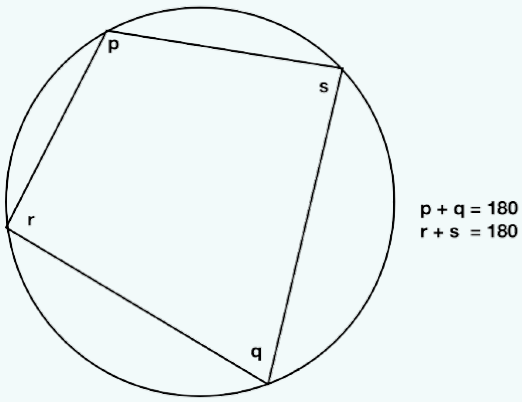
4.9 Cyclic Quadrilaterals

Video Link(s)

Cyclic Quadrilaterals

Theorem 4.9.1 (Properties of a Cyclic quadrilateral)

$$\text{Sum of opposite angles} = 180$$

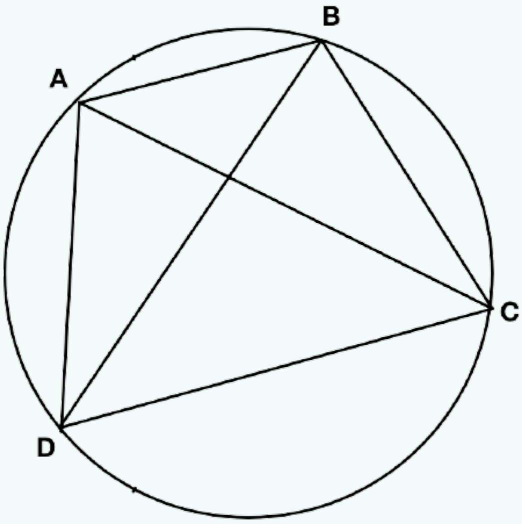


Theorem 4.9.2 (Ptolemy's Theorem)

In a cyclic quadrilateral $ABCD$

$$\text{Product of diagonals} = \text{Sum of the product of both pairs of opposite sides}$$

$$\mathrm{AC} \cdot \mathrm{BD} = \mathrm{AB} \cdot \mathrm{CD} + \mathrm{AD} \cdot \mathrm{BC}$$



Video Link(s)

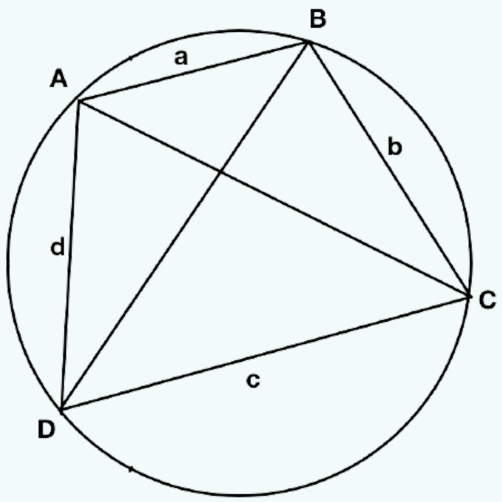
Ptolemy's Theorem

Theorem 4.9.3 (Brahmagupta's Formula)
In a cyclic quadrilateral with side lengths a, b, c, d, the area of the quadrilateral can be found as:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where s is the semiperimeter of the quadrilateral and can be calculated as

$$s = \frac{a+b+c+d}{2}$$



Basically, to find the area of a cyclic quadrilateral

1. Find the perimeter and divide by 2
2. Subtract each of the side lengths from it to get 4 values
3. Multiply your 4 values
4. Take the square root of your product

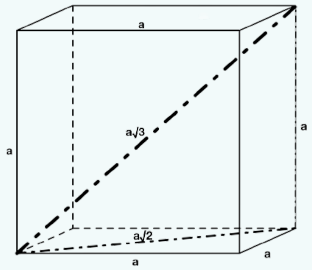
4.10 3D Geometry

4.10.1 Cube

Theorem 4.10.1 (Volume and Surface Area of a cube)

$$\text{Volume of a cube} = \text{side length}^3 = a^3$$
$$\text{Surface area of a cube} = 6 \times \text{side length}^2 = 6a^2$$

$$\text{Length of space diagonal of a cube} = \sqrt{3} \times \text{side length} = \sqrt{3}a$$

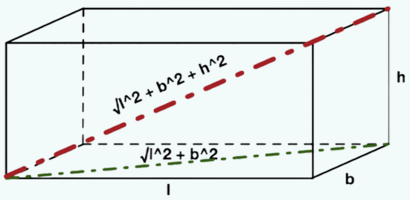


4.10.2 Rectangular Prism

Theorem 4.10.2 (Volume and Surface Area of a rectangular prism)

$$\text{Volume of a rectangular prism} = l \times w \times h = \text{product of all 3 dimensions}$$
$$\text{Surface area of a rectangular prism} = 2(lw + wh + lh)$$

$$\text{Length of space diagonal of a rectangular prism} = \sqrt{l^2 + b^2 + h^2}$$

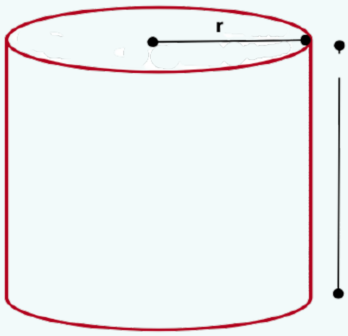


4.10.3 Cylinder

Theorem 4.10.3 (Volume and Surface Area of a cylinder)

$$\text{Volume of a cylinder} = \pi r^2 h$$

$$\begin{aligned}\text{Surface area of a cylinder} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r(r+h)\end{aligned} \tag{4.1}$$



4.10.4 Cone

Theorem 4.10.4 (Volume and Surface Area of a cone)

$$\text{Volume of a cone} = \frac{1}{3}\pi r^2 h$$

which basically means

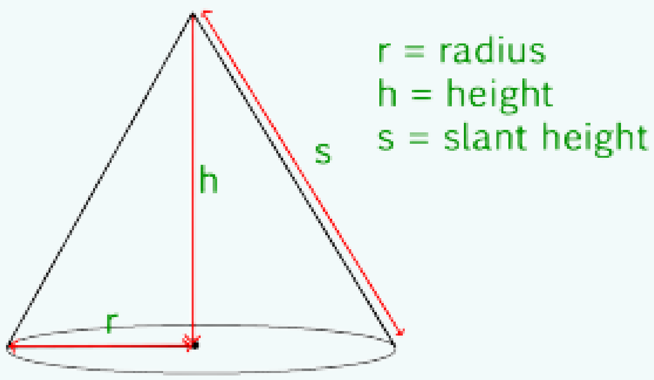
$$\text{Volume of a Cone} = \frac{1}{3}\pi \cdot \text{radius}^2 \cdot \text{height}$$

$$\text{Surface area of a cone} = \pi r^2 + \pi r s = \pi r(r+s)$$

where s is the lateral or slant height

which can also be written as

$$\pi \cdot \text{radius}^2 + \pi \cdot \text{radius} \times \text{slant height}$$



Remark 4.10.5
The slant height s can be calculated by the following formula

$$s = \sqrt{r^2 + h^2}$$

or

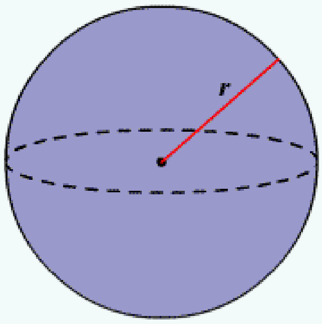
$$\text{slant height} = \sqrt{\text{radius}^2 + \text{height}^2}$$

4.10.5 Sphere

Theorem 4.10.6 (Volume and Surface Area of a sphere)

$$\text{Volume of a sphere} = \frac{4}{3}\pi r^3$$

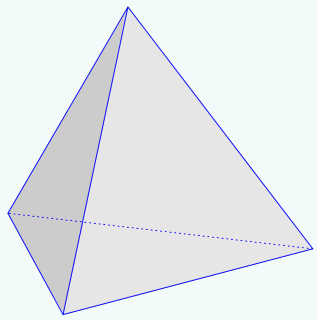
$$\text{Surface area of a sphere} = 4\pi r^2$$



4.10.6 Tetrahedron

Theorem 4.10.7 (Volume of a tetrahedron)

$$\text{Volume of any tetrahedron} = \frac{1}{3} \cdot \text{base area} \cdot \text{height}$$



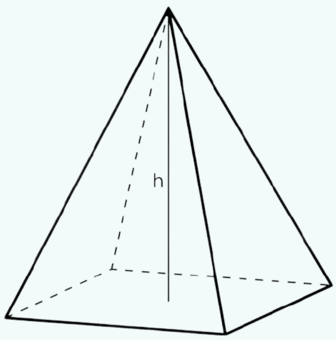
Theorem 4.10.8 (Volume of a regular tetrahedron (all sides equal))

$$\text{Volume of a regular tetrahedron} = \frac{\sqrt{2}}{12} s^3$$

4.10.7 Pyramid

Theorem 4.10.9 (Volume of a pyramid)

$$\text{Volume of any pyramid} = \frac{1}{3} \cdot \text{base area} \cdot \text{height}$$



Theorem 4.10.10 (Volume of a regular pyramid (all sides equal))
When the pyramid has a square base, and all the sides are equal

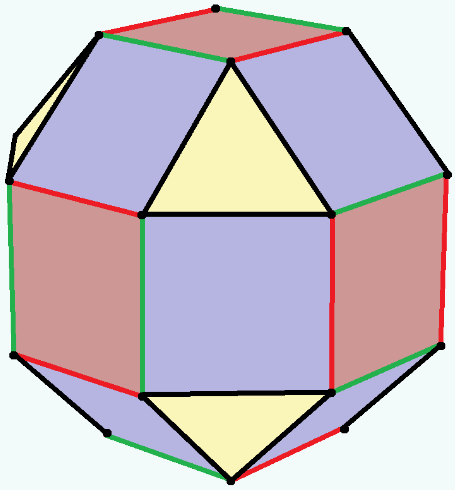
$$\text{Volume of a regular pyramid} = \frac{\sqrt{2}}{6}s^3$$

Concept 4.10.11 (Cavalieri's principle)
If there are two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal

4.10.8 Polyhedron

Theorem 4.10.12 (Euler's Polyhedron Formula)

$$\begin{aligned} \text{Edges} &= \text{Faces} + \text{Vertices - 2} \\ \text{E} &= \text{F + V - 2} \end{aligned}$$



Remark 4.10.13
If you ever forget this theorem, just think of a cube and remember that it has 6 faces, 8 vertices, and 12 edges.

Remark 4.10.14
Note that in a polyhedron, if we know or can get information about the number of faces/edges coming out of 1 point, and the number of vertices shared for every face/edge, we can easily calculate the number of faces/edges from the number of vertices and vice versa.

Video Link(s)
3D Geometry

4.11 Advanced Formulas

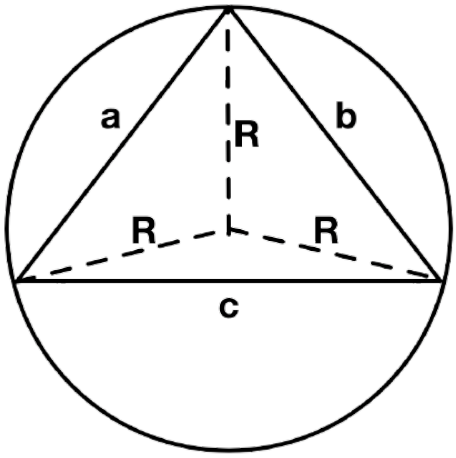
Definition 4.11.1 (Median)**.** A median is a line connecting a point to the midpoint of the opposite side.

Definition 4.11.2 (Centroid)**.** In a triangle, the intersection of all 3 medians in a triangle is the centroid.

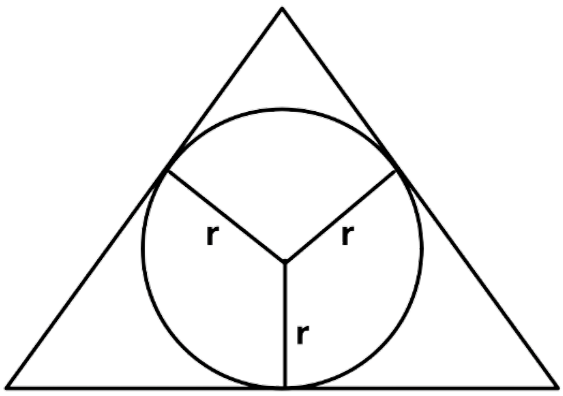
Theorem 4.11.3

The centroid of a triangle is on the median and it is $\frac{2}{3}$ of the way from from one of vertices to the midpoint of the opposite side.

Definition 4.11.4 (Circumcenter)**.** The circumcenter of a triangle is the intersection of all 3 perpendicular bisectors (the line that bisects a segment and is perpendicular to it). This point is also the center of the circumcircle and equidistant from all the three vertices.



Definition 4.11.5 (Incenter)**.** The incenter of a triangle is the intersection of all the angle bisectors. This point is also the center of the incircle, and equidistant from all the three sides.



Theorem 4.11.6

Inradius r of a right triangle:

$$r = \frac{1}{2}(a + b - c)$$

where a and b are the legs of the triangle, and c is the hypotenuse.

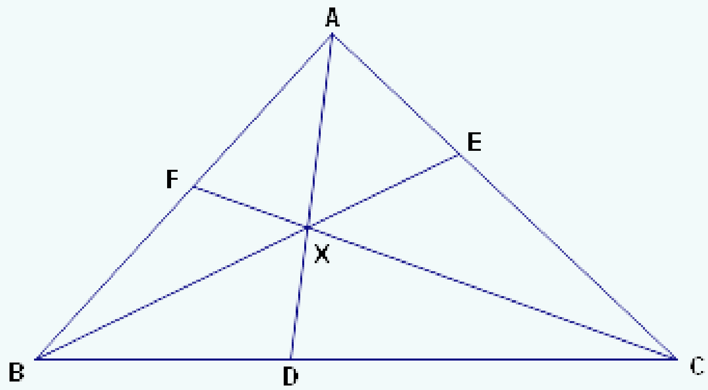
Definition 4.11.7 (Cevian)**.** A cevian is any line from any vertex of a triangle to the opposite side. Medians and angle bisectors are special cases of cevians.

Theorem 4.11.8 (Ceva's Theorem)

In a triangle ABC, and let D, E, F be points on lines BC, CA, AB, respectively. Lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$$

Note: It is not necessary that these cevians lie within the triangle



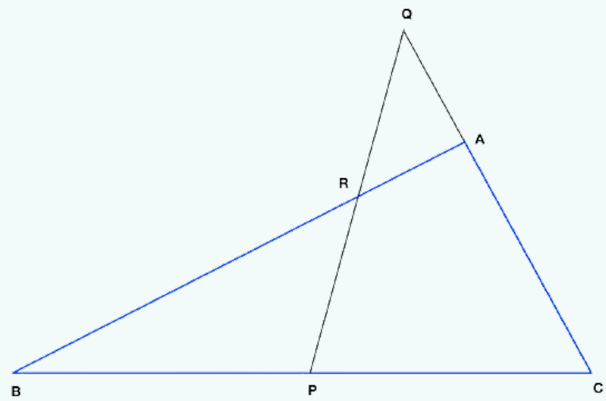
Remark 4.11.9

A way to remember this is that you are going around the triangle multiplying ratios.

Theorem 4.11.10 (Menelaus's Theorem)

If line PQ intersecting AB on $\triangle ABC$, where P is on BC, Q is on the extension of AC, and R on the intersection of PQ and AB, then

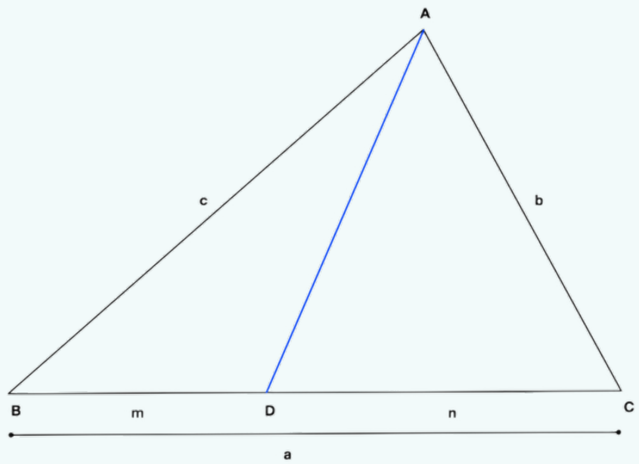
$$\frac{PB}{CP} \cdot \frac{QC}{QA} \cdot \frac{AR}{RB} = 1.$$



Theorem 4.11.11 (Stewart's Theorem)
Given a triangle $\triangle ABC$ with sides of length a, b, c opposite vertices of A, B, C respectively. If cevian AD is drawn so that $BD = m$, $DC = n$ and $AD = d$, we have that

$$man + dad = bmb + cnc$$

.



Remark 4.11.12
A way to remember this is the saying "A **Man** and his **Dad** put a **Bomb** in the **Sink**

Corollary 4.11.13 (Stewart's For Angle Bisector)
If AD is an angle bisector, then $d^2 + mn = bc$.

Note that this follows from Stewart's Theorem and the Angle Bisector Theorem.

Corollary 4.11.14 (Stewart's Theorem For Medians)
If AD is a median, then $d^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2$

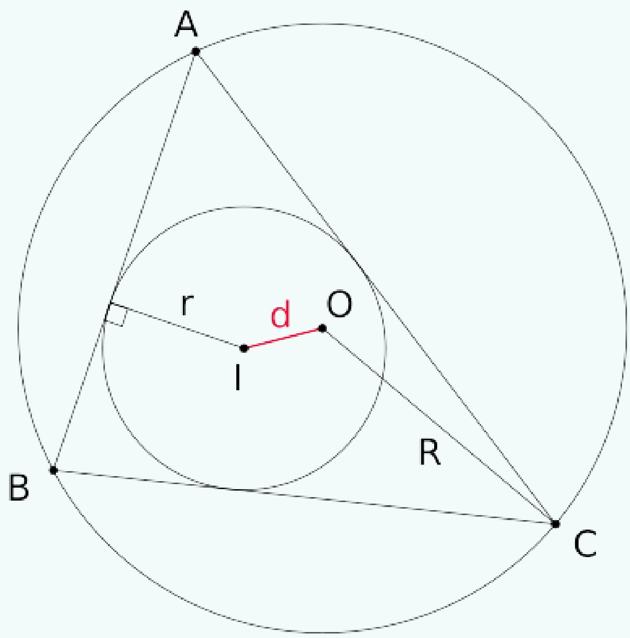
Theorem 4.11.15 (Euler's Geometry Theorem)
Euler's theorem states that the distance d between the circumcenter and incenter of a triangle is given by

$$d^2 = R(R - 2r)$$

or equivalently

$$\frac{1}{R-d} + \frac{1}{R+d} = \frac{1}{r}$$

where R and r denote the circumradius and inradius respectively.



4.12 Coordinate Geometry

4.12.1 Line in Coordinate Plane

Definition 4.12.1 (Equation of a Line)**.** The equation of a line is $ax + by + c = 0$.

Definition 4.12.2 (Slope-Intercept Form)**.** An equation of a line in slope-intercept form is $y = mx + b$

Theorem 4.12.3 (Slope of a line through 2 points)
The slope of a line passing through 2 points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Theorem 4.12.4 (Slope of a line through angle)
The slope of a line with an angle of θ above the x-axis is $\tan(\theta)$

Theorem 4.12.5 (Distance between 2 points)
The distance between 2 points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Theorem 4.12.6 (Point to line formula)
The distance between a point (x_0, y_0) and a line $ax + by + c = 0$ is

$$\frac{|a \cdot x_0 + b \cdot y_0 + c|}{\sqrt{a^2 + b^2}}$$

.

Remark 4.12.7
Be careful not to get the equation of the line confused with $ax + by = c$

Remark 4.12.8
Note that this distance represents the shortest possible distance which would be length of the perpendicular line.

Remark 4.12.9
This formula is a bit confusing so an easy way to remember the numerator is that it's just the equation of the line with the values of the point plugged in as the x and y values

4.12.2 Circle in Coordinate Plane

Theorem 4.12.10 (Equation of a Circle)
A circle with center (a, b) and radius r has equation

$$(x-a)^2 + (y-b)^2 = r^2$$

Theorem 4.12.11 (Equation of a Parabola)
A parabola with vertex (a, b) has an equation of $y = k(x-a)^2 + b$ for any real non-zero constant k.

Theorem 4.12.12 (Equation of an Ellipse)
An ellipse has an equation of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where the length of the 2 radii of the ellipse are a and b

Remark 4.12.13
Note that when $a = b$, it becomes a circle.

Theorem 4.12.14 (Equation of a Hyperbola)
A hyperbola has an equation of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where the the rectangle that the hyperbola can fit inside has a width of a and length of b. A hyperbola consists of 2 symmetric parts that can be reflected across the y-axis.

Chapter 5

Trigonometry

Note: This topic is mainly relevant for AMC 12, but knowing some concepts like Law of Cosines can help make some AMC 10 problems easier to solve.

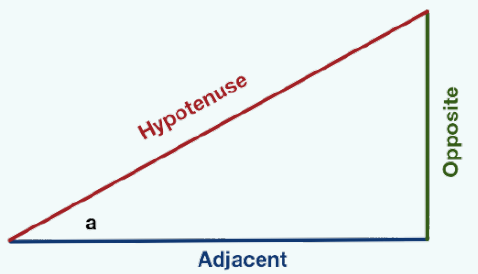
5.1 Trigonometric Identities

Theorem 5.1.1 (Trigonometric Identities)

$$\text{Sine: } \sin(a) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\text{Cosine: } \cos(a) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\text{Tangent: } \tan(a) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sin(a)}{\cos(a)}$$



Remark 5.1.2

To remember the relationships, just use the mnemonics SOH, CAH, TOA:

SOH = Sin is Opposite over Hypotenuse

CAH = Cos is Adjacent over Hypotenuse

TOA = Tan is Opposite over Adjacent

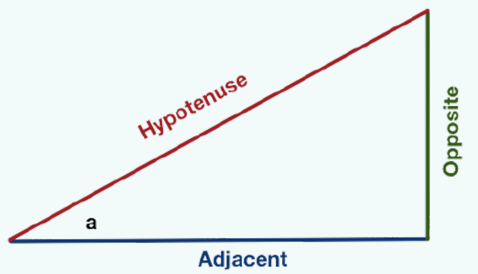
5.2 More Trigonometric Identities

Theorem 5.2.1 (More Trigonometric Identities)

$$\text{Cosecant: } \csc(a) = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{1}{\sin(a)}$$

$$\text{Secant: } \sec(a) = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{1}{\cos(a)}$$

$$\text{Cotangent: } \cot(a) = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{1}{\tan(a)} = \frac{\cos(a)}{\sin(a)}$$



5.3 Important Trigonometric Values

$\cos 0° = 1$	$\sin 0° = 0$
$\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$	$\sin 15° = \frac{\sqrt{3}-1}{2\sqrt{2}}$
$\cos 30° = \frac{\sqrt{3}}{2}$	$\sin 30° = \frac{1}{2}$
$\cos 45° = \frac{\sqrt{2}}{2}$	$\sin 45° = \frac{\sqrt{2}}{2}$
$\cos 60° = \frac{1}{2}$	$\sin 60° = \frac{\sqrt{3}}{2}$
$\cos 75° = \frac{\sqrt{3}-1}{2\sqrt{2}}$	$\sin 75° = \frac{\sqrt{3}+1}{2\sqrt{2}}$
$\cos 0° = 1$	$\sin 0° = 0$
$\cos 90° = 0$	$\sin 90° = 1$
$\cos 120° = -\frac{1}{2}$	$\sin 120° = \frac{\sqrt{3}}{2}$
$\cos 135° = -\frac{\sqrt{2}}{2}$	$\sin 135° = \frac{\sqrt{2}}{2}$
$\cos 150° = -\frac{\sqrt{3}}{2}$	$\sin 150° = \frac{1}{2}$
$\cos 180° = 1$	$\sin 0° = 0$

5.4 Unit Circle Identities

Theorem 5.4.1 (Unit Circle Identities)

$$\sin(-a) = -\sin(a)$$

$$\sin(a) = \sin(180 - a)$$

$$\cos(a) = \cos(-a)$$

$$\cos(a) = -\cos(180 - a)$$

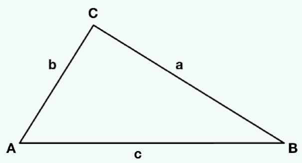
$$\tan(a) = -\tan(180 - a)$$

$$\tan(-a) = -\tan(a)$$

5.5 Area of a Triangle using trigonometry

Theorem 5.5.1 (Area of a Triangle using trigonometry)
In a triangle with side lengths, a, b, c, where the angle between sides a and b is denoted by C

$$\text{Area of the triangle} = \frac{1}{2} \cdot ab \cdot \sin(C)$$



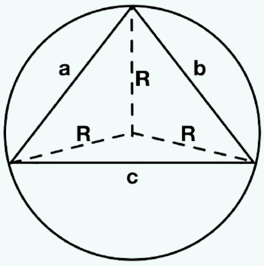
5.6 Law of Sines

Theorem 5.6.1 (Law of Sines)

In a triangle with sides a, b, c and angles A, B, and C where the side a is opposite to the angle A, the side b is opposite to the angle B, and the side c is opposite the angle C, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of the triangle.

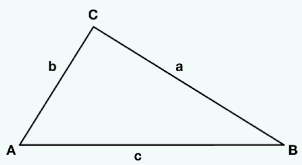


5.7 Law of Cosines

Theorem 5.7.1 (Law of Cosines)

In a triangle with side lengths, a, b, c, where the angle between sides a and b is denoted by C

$$c^2 = a^2 + b^2 - 2ab \cdot \cos C$$

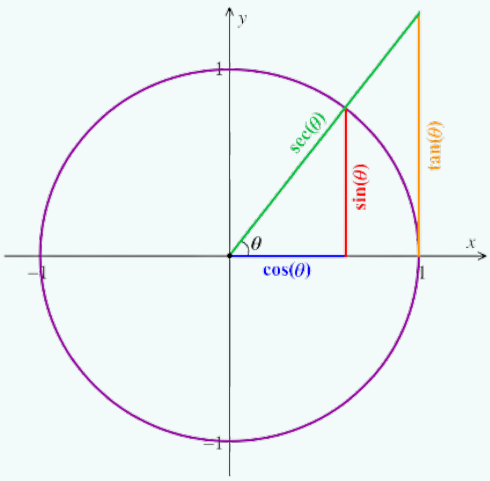


Video Link(s)

Solving area problems using trigonometry

5.8 Pythagorean Identities

Theorem 5.8.1 (Pythagorean Identities)



$$\sin^2(a) + \cos^2(a) = 1$$

$$\tan^2(a) + 1 = \sec^2(a)$$

$$\cot^2(a) + 1 = \csc^2(a)$$

5.9 Double Angle Identities

Theorem 5.9.1 (Double Angle Identities)

$$\sin(2a) = 2\sin(a)\cos(a)$$

$$\cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$$

$$\tan(2a) = \frac{2\tan(a)}{1 - \tan^2(a)}$$

5.10 Addition and Subtraction Identities

Theorem 5.10.1 (Addition and Subtraction Identities)

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

$$\sin(a-b) = \sin(a)\cos(b) - \sin(a)\cos(b)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan a \tan b}$$

$$\tan(a-b) = \frac{\tan(a)-\tan(b)}{1+\tan a \tan b}$$

5.11 Half Angle Identities

Theorem 5.11.1 (Half Angle Identities)

$$\sin\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1-\cos(a)}{2}}$$

$$\cos\left(\frac{a}{2}\right) = \pm\sqrt{\frac{1+\cos(a)}{2}}$$

5.12 Sum to Product Identities

Theorem 5.12.1 (Sum to Product Identities)

$$\sin(a) + \sin(b) = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

$$\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$$

$$\cos(a) - \cos(b) = -2\sin\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$$

5.13 Product to Sum Identities

Theorem 5.13.1 (Product to Sum Identities)

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$$

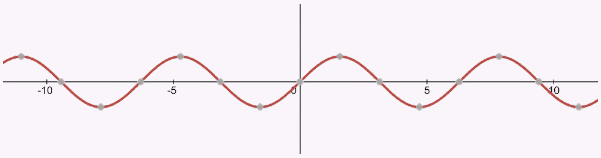
$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$$

Definition 5.13.2. A periodic function is a trigonometric function which repeats a pattern of y-values at regular intervals. One complete repetition of the pattern is called a cycle. The period of a function is the horizontal length of one complete cycle.

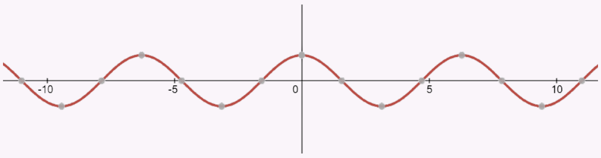
Period of sin, cos, and tan is 2π

5.14 Periods and Graphs of Trigonometric Functions

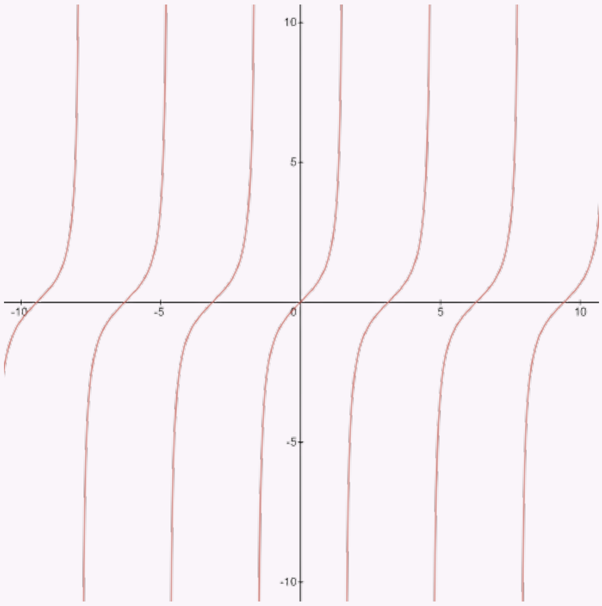
Concept 5.14.1 (Sine Graph)



Concept 5.14.2 (Cosine Graph)



Concept 5.14.3 (Tan Graph)



Remark 5.14.4
Long trigonometric expressions can be evaluated by telescoping, using identities in clever ways, complex number substitutions (see complex numbers section below).

Chapter 6

Logarithms

Note: This topic is mainly relevant for AMC 12.

Video Link(s)

Logarithms

6.1 Basic Definitions

Definition 6.1.1 (Logarithm)**.** A logarithm is the power to which a number must be raised in order to get some other number.

Logarithms are expressed as

$$a = log_b n$$

where b is the base and n is the number.

Basically, we are trying to calculate how many times we need to multiply the base to get the number a, or what power do we need to raise the base to get the number a.

Theorem 6.1.2 (Converting to Logarithm and Exponents)

$$log_x y = a \implies x^a = y$$

.

6.2 Logarithmic Formulas

Theorem 6.2.1 (Important Formulas)

$$\log_a a^r = r$$

$$\log_a bc = \log_a b + \log_a c$$

$$\log_a \frac{b}{c} = \log_a b - \log_a c$$

$$\log_a b^c = c \log_a b$$

$$\log_a b \cdot \log_b c = \frac{\log_b}{\log_a} \cdot \frac{\log_c}{\log_b} = \frac{\log_c}{\log_a} = \log_a c$$

$$\log_b a = \frac{1}{\log_a b}$$

$$\log_b a = \frac{\log_d a}{\log_d b}$$

Remark 6.2.2
This last formula is known as the "Base Change Formula" and is the most useful of them all. Often times in logarithm problems you can just expand out your expression in terms of this formula and simplify the expression to get the answer.

Remark 6.2.3
These formulas are extremely important for working with logarithms and should definitely be memorized.

Remark 6.2.4
If you ever forget which way the sign of these logarithms are, you can just try a small example like $log_{10}100 + log_{10}1000 = log_{10}100,000$ so from here for example you could figure out the sum of logarithms identity.

Theorem 6.2.5 (Advanced Formulas)

$$\log_{a^m} a^n = \frac{n}{m}$$

$$\log_a \left(\frac{1}{b}\right) = -\log_a b$$

$$\log_{\frac{1}{a}} b = -\log_a b$$

Remark 6.2.6
These formulas are less important and aren't necessary for most logarithm problems, but still good to know.

Chapter 7

Complex Numbers

Note: This topic is mainly relevant for AMC 12.

7.1 Basic Definitions

Definition 7.1.1. A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i represents the "imaginary unit". a is the real part of our number, and bi is the imaginary part. Complex numbers are often represented by the variable z.

Definition 7.1.2. $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

Remark 7.1.3
Powers of i cycle every 4 terms, so $i^{4n} = i^4 = 1, i^{4n+1} = i = \sqrt{-1}, i^{4n+2} = i^2 = -1, i^{4n+3} = i^3 = -i$

Theorem 7.1.4 (Adding Complex Numbers)

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Theorem 7.1.5 (Subtracting Complex Numbers)

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Theorem 7.1.6 (Multiplying Complex Numbers)

$$(a + bi) \cdot (c + di) = (ac - bd) + (bc + ad)i$$

Definition 7.1.7 (Real and Imaginary Parts)**.** The imaginary part of a complex number $a + bi$ is b and the real part is a.

Remark 7.1.8
The imaginary part does not include a factor of i.

7.2 Conjugates

Definition 7.2.1. A complex conjugate is found by flipping the sign of the imaginary part of complex number, and is represented as $\bar{z}$.

Theorem 7.2.2 (Finding Conjugates)

$$\overline{z} = \overline{a + bi} = a - bi$$

Theorem 7.2.3 (Multiplying Complex Numbers with their Conjugates)

$$(a + bi)(a - bi) = a^2 + b^2$$

7.3 Complex Roots

Definition 7.3.1. A polynomial of degree n has n roots, and these roots may be complex. For binomials, if our discriminant is negative we have complex roots.

7.4 Complex Plane

Definition 7.4.1. A complex number can also be represented geometrically by expressing $a + bi$ as (a, b) on the Complex plane. The x-axis represents the Real axis and the y-axis represents the Imaginary axis.

Definition 7.4.2. The magnitude of a complex number is represented by $|z|$, and is the distance of a complex number (a, b) from the origin.

Theorem 7.4.3 (Magnitude of a Complex Number)

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$

7.5 Polar Form

Definition 7.5.1. The angle that the positive real axis makes with the ray that connects the origin with a complex number is called the argument of that complex number and is represented by θ.

Theorem 7.5.2 (Argument of a Complex Number)
The argument θ of a complex number $a + bi$ is

$$\tan\theta = \frac{b}{a}$$

.

Definition 7.5.3. The distance between 0 and a complex number is sometimes called the modulus of that complex number and is represented by r.

Theorem 7.5.4 (Modulus of a Complex Number)
The modulus r of a complex number $a + bi$ is

$$r = |a + bi| = \sqrt{a^2 + b^2}$$

.

Definition 7.5.5. Polar form is another way to represent a complex number based on its modulus r and argument θ.

Theorem 7.5.6 (Polar Form)

$$z = a + bi = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$$

Remark 7.5.7
cis θ is just short for $\cos\theta + i\sin\theta$

Remark 7.5.8
Trigonometric ratios tell us that $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$, which we can rearrange to see that $r\cos\theta = a$ and $r\sin\theta = b$. Plugging in these values gives us the polar form formula.

Remark 7.5.9
$\cos\theta + i\sin\theta$ can also be written as $\operatorname{cis}\theta$.

Theorem 7.5.10 (Euler's Formula)
Euler's Formula tells us that

$$\cos\theta + i\sin\theta = e^{i\theta}$$

, which tells us that

$$z = a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

.

Remark 7.5.11
Euler's Identity is a special case of Euler's Formula and tells us that

$$e^{\pi i} = -1$$

.

Theorem 7.5.12 (De Moivre's Theorem)
For a complex number $z = re^{i\theta}$ and a real number n,

$$z^n = (re^{i\theta})^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

Remark 7.5.13
We can use this to evaluate expressions like

$$(3 + 2i)^8$$

much easier because we just convert to polar form and apply De Moivre's Theorem.

Remark 7.5.14
DeMoivre's Theorem is very useful when dealing with complex numbers and exponents.

Theorem 7.5.15 (Rotating a Point)
To rotate a point θ radians counterclockwise, covert a coordinate to its corresponding complex number and multiply it by $e^{i\theta}$. Converting this back to ordered pairs gives us our answer.

Concept 7.5.16
Complex numbers and their relations to circles makes them easy to work with for many geometry problems, especially when dealing with polygons such as equilateral triangles or squares.

How to solve geometry problems using complex numbers:

1. Assign a complex number to 1 or more of the coordinates
2. To find the complex numbers for other points, multiple/divide by $e^{i\theta}$
3. Use the information you have to solve for what you are asked in the problem

Remark 7.5.17
We can also view algebraic complex number problems geometrically.

7.6 Roots of Unity

Definition 7.6.1. Roots of unity are the complex solutions to an equation $x^n = 1$, for some positive integer n. There will always be n solutions to $x^n = 1$.

Theorem 7.6.2 (Roots of Unity)
The set of the nth roots of unity is

$$e^{2k\pi i/n}$$

for

$$k \in \{1, 2, \ldots n\}$$

.

7.7 Complex Numbers for Trigonometry

Theorem 7.7.1 (Sin and Cos values in terms of complex numbers)

$$\sin\theta = \frac{e^{i\pi} + e^{i(180-\pi)}}{2i}$$

$$\cos\theta = \frac{e^{i\pi} + e^{i(-\pi)}}{2}$$

$$\tan\theta = \frac{e^{i\pi} + e^{i(180-\pi)}}{e^{i\pi} + e^{i(-\pi)}} i$$

Remark 7.7.2
By using these substitutions, we can bash out the value of trigonometric expressions easily without clever manipulation of trigonometric identities that would be needed to solve the problem otherwise.

Chapter 8

Additional Techniques and Strategies

8.1 Meta-solving Techniques

Video Link(s)

Meta-solving Techniques

Definition 8.1.1. Meta-solving is finding the answer to a problem without actually solving it.

Remark 8.1.2
These techniques may not work for all problems. These techniques are especially useful when the problem provides answer choices.

Remark 8.1.3 (Meta-Solving Warning)
Don't get too carried away with these techniques to the point where you don't even try to solve the problem legitimately.

Concept 8.1.4 (Engineering Induction)
Engineering Induction is the process of trying and finding the value to small cases and assuming it's true for larger ones.

Steps for Engineering Induction Problems:

1. Try small cases
2. Look for a pattern amongst those small cases (there may not always be one)
3. Assume the pattern can continue for larger cases and find the answer

Remark 8.1.5

We can try to apply engineering induction when we see the values in the problem seem hard/impossible to compute.

Concept 8.1.6 (Looking for unique properties of numbers)

Rather than computing the exact answer, we can compute unique properties of your answer choices so that you can eliminate all answer choices that don't satisfy the property and so that you will be left with 1 answer choice (or possibly more in which case you can just guess from the remaining ones)

Some unique properties you can look for in your answer choices and try to compute are

- Units Digit
- Last 2 digits
- Parity (Even, Odd)
- Perfect square or not one
- Prime/composite
- Modulus (remainders when divided by 3, 4, 5, etc.)
- Denominators of common fractions (or what they must divide)
- Multiples/Factors of numbers
- etc.

Remark 8.1.7

These last 2 properties are especially useful in combinatorics problems as you can easily find numbers you have to multiply with each other to get your answer.

Concept 8.1.8

Look for the option choices that are the "odd one out" or that are different from all others

- Look for outliers (primes, large/small numbers, odd/even numbers, powers of 2, etc)

Concept 8.1.9 (Trying all the Option choices)

In some problems, you can

- Try all the option choices into the conditions in the problem
- Look at the conditions in the problem and see which of the option choices could work
- etc.

After doing so, you will either have a better guess or exact answer.

Concept 8.1.10 (Elimination of Option Choices)

Tying closely to the previous technique, you can also usually eliminate option choices based on

- Unique properties of Numbers (See Above)
- How large or small the number must be

Remark 8.1.11

We would recommend guessing ONLY if you can narrow it to 2 or 3 option choices.

Concept 8.1.12 (Estimation of Answer)

Often times in problems (especially geometry) you can easily find an approximate answer and see which of the option choices most closely matches what you got.

In geometry, a common strategy to do so is to mark out areas approximately equal to those of areas you know.

Concept 8.1.13 (Using Freedom in Problems)

Assuming facts when you have freedom in the problem statement can be very useful.

Essentially, as long as the problem is not telling you ”this fact is not true” (so, whatever assumption you want to make will satisfy the problem’s conditions) you can assume the fact is true to simplify your problem and make it really easy to solve.

For example, if you are asked to find some universal ratio in a triangle and you aren’t specifically told that the triangle isn’t equilateral, you can just assume the triangle is equilateral and solve the remaining problem from there.

Remark 8.1.14

Make sure not to assume false information! Be very careful that your assumption can be true.

In our previous example, if we were told the triangle had 2 sides of length 7 and 8, then our assumption would be false, so it wouldn’t work then.

8.2 Silly Mistakes

Silly mistakes are very common and can really lower your score on the AMC 10/12. Here are some tips on how to avoid the different kinds of silly mistakes:

Concept 8.2.1 (Avoiding Arithmetic Errors)
A good way to avoid arithmetic errors is to

- Do your computation 2 ways (e.g. If you have to do $87 \cdot 93$, you can multiply them with 87 on the top and with 93 on top)
- Be more organized, and write more steps

Concept 8.2.2 (Avoiding Mathematical Errors)
An easy way to avoid mathematical errors is to

- Do your work neatly!
- Make boxes for each problem on scratch paper per problem
- Don't skip steps
- Check your work, following the tip above will make it easier to do so
- Do your steps methodically
- Try to substitute your answer back into the problem (if you can)
- Try an alternate solution to confirm your answer
- Estimate what the answer has to be, and see if your answer is close to what your estimate is

Concept 8.2.3 (Avoid Reading Errors)

Reading the problem wrong is one of the most common mistakes. Often times, you might forget about important key words like

- inclusive, except
- even, odd
- prime, composite
- integer, natural, real, complex

Some strategies to avoid them are

- After solving the problem, reread the question part of the problem to make sure you are answering what the question asks for!
- Underline key words while reading (that's probably gonna be hard this year with the test being online, but as an alternative you can take note of the important words on your scratch paper)

Concept 8.2.4 (Avoid Missing a Final Step Errors)

Sometimes in problems, you might be so caught up in moving forward in the test that you might forget an important step at the end.

For example, in a problem you might think "I'll multiply by 5 to whatever answer I get" and then you find that answer but forget to multiply by 5. A way to avoid this is:

- Write "Remember ..." big and bold on your scratch or the question paper

Remark 8.2.5

A very common reading mistake is getting confused between the words non-negative and positive. Remember, non-negative includes 0 while positive doesn't!

Concept 8.2.6 (Avoid Making False Assumptions)

Often times, you might just think something is true and assume it's true for the rest of the problem, when really it was false. Proving all your assumptions can be too time consuming. However we recommend at least seeing some sort of reasoning for why your assumption should be correct (unless of course you are using one of the meta-solving techniques).

Concept 8.2.7 (Avoid Casework Errors)

Many times, in casework problems you might

- Undercount possibilities
- Overcount possibilities
- Miss edges/extreme cases

Some strategies to avoid them are

- Be methodical in your casework
- Make sure all your cases work
  - An easy way to do this is just to try a few examples in your case to see if they actually work
  - Especially, make sure to check if extremes work
  - Make sure all your cases are disjoint and that you are not overcounting anything that's common between the cases
  - Make sure your cases cover all possibilities that the problem asks for
  - Solve the problem in multiple ways (for example, by both casework and complementary counting)

Remark 8.2.8

I know that some of these strategies may take a lot of extra time to follow so we recommend analyzing how you are making silly mistakes and from there see which strategies you will want to follow.

8.3 Other Strategies To Maximize Your Score

Concept 8.3.1 (Plan Your Time)

Make sure to plan your time.

- Questions 1-10 are generally easy, 11-20 are medium, 21-25 are hard
- Sometimes one of the early questions can be hard or bashy, or a later question can be easy
- Don't get stuck on a question. Move on and come back to it later.
- Star any question you are unsure about but you feel you can solve it, or any questions that you solved but are not confident of your answer
- Budget your time properly
- Leave enough time for last 10 problems
- Leave some time to review starred problems and check your work

Concept 8.3.2 (Guessing Strategies)

If you want to guess, make sure to keep these things in mind:

- You get 1.5 points for leaving a question blank, so if you don't know how to solve the problem, just leave it blank
- Try using meta-solving strategies above
- If you can narrow down the choices to 2-3 options, only then make an educated guess

Concept 8.3.3 (Test Day Strategies)
Here are some test day strategies:

- Don't try to cram or study on the day before the test
  - Just review a few formulas or strategies from this book
- Be relaxed
  - Get a good night's sleep
  - Eat a healthy meal
  - Meditate
  - Eat dark chocolate
  - Listen to music
  - Or whatever else makes you feel relaxed

Concept 8.3.4 (Problem Solving Strategies)
Try to remember these problem solving strategies:

- When you are stuck, try to use the information in the problem you haven't used yet
- When solving a problem, think of what technique would likely be at play
- Don't get too stuck with one approach to a problem, move on, and come back to the problem with a fresh perspective
- In problems that seem complex, try small cases to look for a pattern that can allow you to figure out how to approach the problem and what patterns may exist (this is similar to engineering induction)

Concept 8.3.5 (AIME Qualification)
To qualify for AIME, based on recent cutoff scores, you will need a score of at least 105. However you should aim for 110+ to be safe. For this, you'll need a 17-0-8 split (right-wrong-skipped), a 18-5-2 split, a 19-6-0 split or something better.

Concept 8.3.6 (USAJMO/USAMO Qualification)
To qualify for USAJMO/USAMO, you'll probably need a score of at least 120 to have a good chance (of course this also depends heavily on your AIME score).

Remark 8.3.7
Another strategy for the AMC 10/12B is that if you are expecting a AIME qualifying score on the AMC 10/12A, then you should try to solve more problems and take a "higher risk higher reward" approach for the AMC 10/12B, however if you didn't do good on the A, you should take a safe path.

Remark 8.3.8
Last, but most important, don't stress out too much about how you will do! It's just a math contest, and you'll probably have many more opportunities in the future.

Good luck to you on your math competitions. We hope you found this book useful! We really appreciate your feedback and can be reached at omegalearn.info@gmail.com. Thanks!

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