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SOME COMMON FIXED POINT THEOREMS IN COMPLETE METRIC SPACE VIA WEAKLY COMMUTING MAPPINGS

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Abstract

"The purpose of this article is to prove some common fixed point theorem using two and four pair of weakly commuting mappings in complete metric space using the result of R.P.Pant[1], Adrian Constantin [3], Shin-sen change[2], H.K.Pathak, Y.J.Cho and S.M.Kang[8], S.Sessa[9] and as a special case".

Keywords: Common Fixed Point, weakly Commuting Mapping, R-weak commutative Complete Metric Space.

I. INTRODUCTION

Weakly commuting mapping is turning point in the fixed point theory when it was formed by Salvatore Sessa [9] and Jungck [12]. The study of some common fixed points of mapping had satisfying certain contractive type conditions which has been the powerful research field in mathematics when Banach introduced its Banach Contraction principle [7].

Shin-Sen Chang[2], S. Sessa & B. Fisher[13] proved some common fixed point theorems for commuting mapping generalizing the Banach contraction principle[7] also Sessa[9] used the notation of Weakly commuting map and the term Compatible mapping in order to generalize the concept of weakly commuting map used by Jungck[12].

All the results on fixed point theorems of weakly commuting mappings gives a new impetus of studying common fixed points of mapping which satisfies some contractive type conditions. In 1986 Jungck [12] obtain weak commutative pair of mappings is compatible but the converse is not true. Adrian Constantin [3] had proved common fixed point theorems involving two pairs of weakly commuting mapping on complete metric space and two fixed point theorems in non-complete metric space.

R-weakly commutativity redefining as occasionally weak commutativity and in recent years the concept of conditionally commuting maps had introduced by Pant & R Pant [15], that if X is complete metric space with ɗ and S and T be two self mapping then it is called conditionally commuting if they commutes on a nonempty subset of the set of coincidence points whenever the set of coincidence is non empty.

1. Preliminaries and notations

In this paper *(X, d)* be a complete metric space [1] and let *S* and *T* be two self mappings of *X*, then *S* and *T* is said to be weakly commuting if the following condition is satisfied

$$d\left(STx, TSx\right) \le d\left(Sx, Tx\right) \text{ for all } x \text{ in } X.$$

Definition 2.1:- Let *X* be complete metric space with ɗ, and S,T be self mapping of *X*, then *S,T* called *R*-Weakly commuting if there exist a positive real no. *R* such that

$$d\left(STx, TSx\right) \le R\, d\left(Sx, Tx\right) \text{ for all } x \text{ in } X.$$

S and *T* are said to be *R*-Weakly commuting if $R > 0$.

Remark 2.2:- Two commuting mappings are weakly commute but two weakly commuting mappings do not necessarily commute [11]. In 1992 R.P.Pant [1] proved two common fixed point theorems for a pair of mappings under the consideration of *R*-weakly commutivity.

Remark 2.3:-Weak commutativity will be *R*-weak commutativity but *R*-weakly commutativity will be weak commutativity if and only if $R \leq 1$.

Definition 2.4:- In a metric space *(X, d)* two self mapping *S* and *T* are said to be Compatible if $d(STx_k, TSx_k) = 0$ for some limit *k*, where $\{x_k\}$ is a sequence in , such that

$$Sx_k = Tx_k = P \text{ for limit } k \text{ and } P \text{ is in } X.$$

It is clear from the above definition that *S* and *T* are non-compatible if there exist at least one sequence $\{x_k\}$ such that

$$lim_{k\to\infty} SXk = lim_{k\to\infty} TXk = P \quad \text{for some } P \in X.$$

Remark 2.5[8]**:-** Compatibility has been point wise *R*-weakly commutative.

Hence we can say that compatibility maps commutate their coincidence point, but point wise *R*-weakly commuting maps need not be compatible.

In 1997 H.K.Pathak, Y.J.Cho and S.M.Kang[8] gave an analogue of *R*-weak commutative by introducing the concept called *R*-weak commutativity of type *(Aj)*.

Definition 2.6:- Two self-mapping *S* and *T* in a metric space *(X, d)* are said to be *R*-weakly commuting of type *(Aj)* if there exist a positive real no. *R* satisfying the condition such that

$$d(STx, TTx) \leq R\, d(Sx, Tx) \text{ for all } x \in X$$

Again in 2002 M. Aamri, D.El Moutawakil [5] defines the property *(E.A)* and generalized the notation of non-compatible maps.

Definition 2.7[18]**:-** Let *X* be complete metric space with *d* and *S,T* be self mapping then *S* and *T* satisfy the property *(E.A)* if there exist a sequence $\{x_k\}$ such that

$$lim_{k\to\infty} SXk = lim_{k\to\infty} TXk = P \quad \text{for some } P \in X.$$

Example 2.8:- let X = $[0, \infty)$ and the mapping $S, T : X \to X$ such that

$$S_k = \frac{K}{7} \quad and \quad T_k = \frac{6K}{7} \quad \text{for all } k \in x$$

Let the sequence $X_k = \frac{1}{k}$ it is clear that

$$lim_{k\to\infty} SXk = lim_{k\to\infty} TXk = 0$$

Hence S and T satisfy the *(E.A)* property.

Theorem 2.1 Let *(X, d)* be a metric space and let *S* and *T* be two continuous self maps in which f is a proper map, then *S* and *T* are compatible if and only if $S_k = T_k$ implies that $ST_k = TS_k$.

Theorem 2.2:- Let *S* and *T* be two maps of a metric space *(X, d)* into itself satisfying

1) $d(Sx, Ty) \leq \rho\,(d((x, y), d(x, Sx), d(y, Ty))$ for all $x, y \in X$ and $\rho \in G$.
2) There exist a point $u \in X$ so that *S* is continuous at *u* and *T* is continuous at the point *Su*.
3) There exist a point $x \in X$ such that the sequence $\{(T0S)^n(x)\} = \{(TS)^n(x)\}$ has a subsequence $\{(TS)^{ni}(x)\}$ on veering to *u*.

Then $u^1 = Su$ is the unique common fixed point of *T* and *S*.

Corollary 2.1:- Let *S* and *T* are two continuous self maps of real no. such that *S* and *T* are strictly increasing. If both map having common fixed point then *S* and *T* are compatible.

Theorem 2.3:- Let *(X, d)* be a complete metric space and let *u*, *v* be *R*-weakly commuting self mappings of *X* satisfying

$$d(ux, uy) \leq \delta\, d(vx, vy) \text{ for all } x, y \in X.$$

Where, $\delta : R_+ \to R_+$ is a continuous function such that

$$\delta(t) < t \quad , \text{ for each } t > 0.$$

If *u(x)* is subset of *v(x)* and if either *u* or *v* is continuous then *u* and *v* have a unique common fixed point in *X* [1].

Theorem 2.4:- Let *u*, *v* be *R*-weakly commuting self-mappings of *X* in a metric space *(X, d)* satisfying the condition such that given $\gamma > 0$ there exist $h(\gamma) > 0$

1) $\gamma \leq d(vx, vy) < \gamma + h \Rightarrow d(ux, uy) < \gamma$
2) $ux = uy$ whenever $vx = vy$.

If $u(x)$ subset of $v(x)$ and if either u or v is continuous then u and v have a unique common fixed point in X.

Example 2.9:- let (X, d) be a metric space where $X = \{0, 1, 1/2, 1/2^2 \ldots\}$ with

$$d(x, y) = |x-y| \text{ for all } x, y \in X.$$

Define the mappings $u : x \to x$ and $v : x \to x$ such that,

$u(0) = 1/2^2$, $u(1/2^k) = 1/2^{k+2}$ and $v(0) = 1/2$, $v(1/2^k) = 1/2^{k+2}$

For $k = 0, 1, 2, 3,$ --------- and (X, d) is complete

$v(x) = \{ ½, 1/2^2, 1/2^3, \text{-----}\}$ contains $\{1/2^2, 1/2^3, \text{-----}\} = u(x)$

Since u and v commute on X they are R-weakly commuting for $R > 0$.

Define $\delta(t) = 1/2\, t$ for all $t > 0$,

u and v both are not continuous at 0. Hence we have

$d(u(0), u(1)) = |¼ - ¼| = 0$

$d(u(0), u(1/2)) = |1/4 - 1/8| = 1/8 = ½ \times ¼$

$= \delta(d(v(0), v(1/2)))$

$d(u(0), u(1/2^2)) = |1/4 - 1/16| = 3/16 = 1/2 \times 3/8$

$= \delta(d(v(0), v(1/2^2)))$

And so on…………

And for $x = 1/2^k$ and $y = 1/2^l$ $(k, l = 0, 1, 2, 3, \text{------})$

We have

$d(ux, uy) = d(u(1/2^k), u(1/2^l)) = |1/2^{k+2} - 1/2^{l+2}|$

$= ½\, |1/2^{k+1} - 1/2^{l+1}| = \delta(d(v(1/2^k), v(1/2^l)))$

$= \delta(d(vx, vy))$.

Hence all conditions are satisfied except continuity of either u or v, but neither u nor v has a fixed point in X.

Example 2.10:- Let d be a usual metric on X, let $X = [0, 1)$ and u, v be self mappings on X such that $ut = 2t\text{-}1$, $vt = t^2$ for all t in X then for any t in X

$$d(uvt, vut) = 2(t-1)^2, \quad d(ut, vt) = (t-1)^2$$

$$i.e.\ d(uvt, vut) = 2\, d(ut, vt)$$

Therefore mappings u and v are R-weakly commuting with $R = 2$ but are not weakly commuting.

Theorem 2.5[3]**:-** let S and U be weakly commuting mappings and let T, V be weakly commuting mappings of a complete metric space (X, d) into itself satisfying the inequality

$$d(Sx,Ty) \le \delta(d(Ux,Vy), d(Ux,Sx), d(Vy,Ty), d(Ux,Ty), d(Sx,Vy)) \forall x, y \in X$$

Where δ *is in* $\mathfrak{I}$ *(set of real and continuous function)*. If the range of U contains the range of T and the range of V contains the range of S and if one of S, T, U and V is continuous then S, T, U and V have a unique common fixed point μ and μ is the unique fixed point of S and U of T and V [3].

Theorem 2.6[6]**:-** let S, T, U and V be four weakly commuting mappings and let T and V be weakly commuting mappings of a complete metric space (X, d) into itself satisfying

$$d(Sx,Ty) \le \delta(d(Ux,Vy), d(Ux,Sx), d(Vy,Ty)) \forall x, y \in X$$

Where δ *is in G (subset of* $\mathfrak{I}$*)*. If the range of U contain the range of T and the range of V contains the range of S and in one of S, T, U and V is continuous then S, T, U and V have a unique common fixed point μ, further μ is unique common fixed point of S, T, U and V.

Theorem 2.7:- let S and U be weakly commuting mappings of a complete metric space (X, d) into itself satisfying the inequality

$$d(Sx,Ty) \le \max\{(cd(Ux,Vy), cd(Ux,Sx), cd(Vy,Ty), ad(Ux,Ty), bd(Vy,Sx))\} \forall x, y \in X$$

Where a, b, c are real numbers such that, $0 \le c < 1$, $0 \le a+b < 1$ and $(\max\{a/(1-a), b/(1-b)\} < 1$.

If the range of U contains the range of T and the range of V contains the range of S, and if one of S, T, U and V is continuous then S, T, U and V have a unique fixed point in μ. Further μ is the unique common fixed point of S, U, T and V.
Moreover we also used the following lemma.

Lemma 2.1[2]**:-** Let a map A satisfies $A:[0,\infty)\rightarrow[0,\infty)$ be right continuous and non-decreasing and for any real no. J belongs to $[0,\infty)$ then there exist a real no $q(J)$ in $[0,\infty)$ such that

1) $q(J)$ is upper bounded $\{q \in [0,\infty) : q < r + A(q)\}$
2) $\lim_{k\rightarrow\infty} A^k [q(r)] = 0$
3) $\lim_{k\rightarrow\infty} A^k [q] = 0$ for every $q > 0$.
4) For any non-negative real no. sequence $\{q_n\}$ we have,
   $q_{k+1} \leq A(q_k)$, for $k = 1, 2, 3,$----
   then, $\lim_{k\rightarrow\infty} q_n = 0$.

Also we see some other useful theorems for the commuting mappings with corollaries.

Corollary 2.2:-
Let S,T,U and V be self mappings of a complete metric space (X, d) and S & T are continuous, suppose $US = SU$ and $VT = TV$, $Ux \subset Tx$ and $Vx \subset Sx$. If there is a function α satisfying α 1 and α 2 where $a+b = 2$ and if for all $x,y \in X$, the inequality (2) of above theorem holds, then each of the pair (U,S) and (V,T) has a unique common fixed point and these two points coincide.

Theorem 2.8[3]**:-** Let (X, d) be a metric space and T be a map of x into itself such that,

i) $d(Tx,Ty) \leq g(d(x,y), d(x,Tx), d(y,Ty)) \forall x,y \in X$, Where $g \in G$.
ii) T is continuous at a point $u \in X$.
iii) There exist a point $x \in X$ such that the sequence of iterates $\{T^n(x)\}$ has a Subsequence $\{T^{n_i}(x)\}$ on veering to u. then u is the unique fixed point of T.

Theorem 2.9[5]**:-** Let (X, d) be a metric space and let S and T be two weakly compatible self mappings such that

1) S and T satisfying the $(E.A)$ property
2) $d(Tl,Tm) < \max\{d(Sl,Sm),[d(Tl,Sl)+d(Tm,Sm)]/2,[d(Tm,Sl)+d(Tl,Sm)]/2\} \forall l \neq m \in X$
3) $Tx \subset Sx$

If Sx or Tx is a complete subspace of X then T and S have a unique common fixed point.

Theorem 2.10[5]**:-** In a metric space (X, d) let S,T,U and V be self mappings such that

1) (S, V) and (T, U) are weakly compatibles.
2) $d(Sx,Ty) \leq F[\max\{d(Vx, Uy), d(Vx, Ty), d(Ux, Ty)\}]$ for all $(x, y) \in X^2$.
3) (S, V) or (T, U) satisfies the property (E.A).
4) Sx is the subset Ux and Tx is subset of Vx.

If the range of one of the mappings S, T, U and V is complete subspace of X then S, T, U and V have a unique common fixed point.

Corollary 2.3:- Let T, U and V be the self mappings in a metric space (X, d) satisfying the following conditions such that

1) (T, V) and (U,v) are weakly compatibles.
2) Tx subst of Vx and Ux subset of Vx.
3) $d(Tx,Uy) \leq F[\max\{d(Vx, Vy), d(Vx, Uy), d(Vy, Uy)\}]$ for all $(x, y) \in X^2$.
4) (T, V) or (U, V) satisfies the property (E.A). if the range of the mappings T,U or V is complete subspace of X, then T,U & V have a unique common fixed point.

II. THE MAIN RESULTS

Followings are the main results of this paper using the special cases of R.P.Pant[1], Adrian Constantin [3], Shin-sen change[2], H.K.Pathak, Y.J.Cho and S.M.Kang[8], S.Sessa[9].

Theorem 2.11[4]**:-** Let S, T, U and V be four self mappings of X such that

i) $T(x) \subset U(x)$ and $S(x) \subset V(x)$

ii) $d(Sx,Ty) \le \alpha(d(Ux,Vy), d(Ux,Sx), d(Vy,Ty), d(Ux,Ty), D(Vy,Sx)) \forall x,y \in X$

If one *S, T, U* and *V* is continuous and *S* and *T* weakly commute respectively with *U* and *V*, then *S, T, U* and *V* have a common fixed point μ.

Furthermore μ is the unique common fixed point of *S* and *U* and *T* and *V*.

Proof: - let x_0 be an arbitrary point in *X*, to prove (i) condition define a sequence

$\{Sx_0, Tx_1, Sx_2, \text{---------------------} Sx_{2k}, Tx_{2k+1}, \text{---------------}\}$

Since by $Sx_{2k} = Vx_{2k+1}$, $Tx_{2k+1} = Tx_{2k+2}$ *for k = 0, 1, 2, ----------*

As we know the result in[14] the sequence i) is a Cauchy sequence. By completeness of X the sequence i) converges to a point μ in X, which is also the limit of the subsequence of i) given by

$\{Sx_{2k},\} = \{Vx_{2k+1}\}$ and $\{Tx_{2k-1}\} = \{Ux_{2k}\}$

Suppose first *U* is continuous then the sequence $\{USx_{2k}\}$ and $\{U^2Sx_{2k}\}$ converges to $U\mu$

Since s weakly commutes with *U* & α is non decreasing then we have

$d(Sx_{2k}, Tx_{2k+1}) \le \alpha\, d(U^2Sx_{2k}, Vx_{2k+1}), d(U^2Sx_{2k}, SUx_{2k}), d(Vx_{2k+1}\, Tx_{2k+1}), d(U^2x_{2k}, STUx_{2k+1}),$
$d(Vx_{2k+1}, SUx_{2k})$

$\le \alpha\, (d(U^2x_{2k}, Vx_{2k+1}), d(U^2x_{2k}, SUx_{2k}), d(Vx_{2k+1}, Tx_{2k+1}), d(U^2x_{2k}, Tx_{2k+1}), d(Vx_{2k+1}, USx_{2k})$
$+ d(USx_{2k}, SUx_{2k}))$

$\le \alpha\, (d(U^2x_{2k}, Vx_{2k+1}), d(U^2x_{2k}, SUx_{2k}), d(Vx_{2k+1}, Tx_{2k+1}), d(U^2x_{2k}, Tx_{2k+1}), d(Vx_{2k+1}, USx_{2k})$
$+ d(Sx_{2k}, Ux_{2k}))$

Let $k \to \infty$ and invoking α is upper semi continuity, we have

$$d(U\mu,\mu) \le \alpha(d(U\mu,\mu), 0, 0, d(U\mu,\mu), d(U\mu,\mu)$$
$$\le \gamma(d(U\mu,\mu)$$

By condition (3 & 4) of lemma 3.1, we get $U\mu = \mu$ and using that result

$$d(S\mu, Tx_{2k+1}) \le \alpha(d(U\mu, Vx_{2k+1}),\ d(U\mu, S\mu), d\left(Vx_{2k+1}, Tx_{2k+1}\right), d(U\mu, Tx_{2k+1}), d(Vx_{2k+1}, S\mu))$$

Taking $n \to \infty$ we get

$$d(S\mu,\mu) \le \alpha(0, d(\mu, S\mu), 0, 0, d(\mu, S\mu))$$
$$< \ \le \gamma(d(S\mu,\mu))$$

By condition (3&4) of lemma 3.1, we have $S\mu = \mu$. Since is in *Sx* subset of *Vx* there exist a point ξ in *X* such that $V\xi = \mu$

By (2) we get,

$$d(\mu, T\xi) = d(S\mu, T\xi) \le \alpha(U\mu, V\xi), d(U\mu, S\mu), d(V\xi, T\xi), d(U\mu, T\xi), d(V\xi, S\mu))$$
$$= \alpha(0, 0, d(\mu, T\xi), d(\mu, T\xi), 0) < \gamma(d(U\mu, T\xi)$$

By condition (3&4) of lemma 3.1, we get $T\xi = \mu$.

As *T* and *V* weakly commute we have,

$$d(T\mu, V\mu) = d(TV\xi, VT\xi) < d(V\xi, T\xi) = d(\mu, \xi) = 0$$

Which gives, $T\mu = TV\xi = VT\xi = V\mu$?

Hence from (2)

$$d(\mu, T\mu) = d(S\mu, T\mu) \le \alpha(d(U\mu, V\mu), d(U\mu, S\mu), d(V\mu, T\mu), d(U\mu, T\mu), d(V\mu, S\mu))$$
$$= \alpha(d(\mu, T\mu), 0, 0, d(\mu, T\mu), d(\mu, T\mu)) < \gamma(d(\mu, T\mu)$$

$\therefore\ T\mu = \mu$ ($\because$ By condition (3&4) of lemma 3.1)

Now let us suppose that the mapping S is continuous, so that $\{S^2x_{2k}\}$ and $\{SUx_{2k}\}$ converges to Sμ.

Since S and U weakly commute it follows as above that the sequence {*USx2k*} converges to the point Sμ. Since S weakly commutes with U and α is non-decreasing then using (2) we get,

$$d(S^2x_{2k}, Tx_{2k+1}) \le \alpha(d(USx_{2k}, Vx_{2k+1}), d(USx_{2k}, S^2x_{2k}), d(Vx_{2k+1}, Tx_{2k+1}), d(USx_{2k}, Tx_{2k+1}), d(Vx_{2k+1}, S^2x_{2k}))$$

Taking $n \to \infty$ we get,

$$d(S\mu,\mu) \le \alpha(d(S\mu,\mu), 0, 0, d(S\mu,\mu), d(S\mu,\mu))$$
$$\le \gamma(d(S\mu,\mu))$$

By condition (3 & 4) of lemma 3.1, we have $S\mu = \mu$

Once again there exist a point ξ in X such that Vξ = μ, thus we get $d(S^2x_{2k},T\xi)\le\alpha(d(USx_{2k},V\xi),d(USx_{2k},S^2x_{2k}),d(V\xi,T\xi),d(USx_{2k},T\xi),d(V\xi,S^2x_{2k}))$ Let $n\to\infty$ it follows that,

$$d(\mu,T\xi)\le\alpha(0,0,d(\mu,T\xi),d(\mu,T\xi),0)$$
$$<\gamma(d(\mu,T\xi)).$$

By condition (3&4) of lemma 3.1, we have $T\xi = \mu$

Since T and V weakly commute, it again follows as above that $T\mu = V\mu$.

$\therefore\ d(Sx_{2k},T\mu)\le\alpha(d(USx_{2k},V_k),d(Ux_{2k},Sx_{2k}),d(V\mu,T\mu),d(Ux_{2k},T_k),d(V\mu,Sx_{2k}))$

Let $n\to\infty$ it follows that,

$$d(\mu,T\mu)\le\alpha(d(\mu,T\mu),0,0,d(\mu,T\mu),d(\mu,T\mu))$$
$$<\gamma(d(\mu,T\mu))$$
$$\therefore T\mu=\mu=V\mu.$$

The point μ is therefore in the range of V and since the range of U contains the range of T, there exist $\chi \in X$ such that $U\chi = \mu$.

$$\therefore d(S\chi,\mu)=d(S\chi,T\mu)$$
$$\le\alpha(d(U\chi,V\mu),d(U\chi,S\chi),d(V\mu,T\mu),d(U\chi,T\mu),d(V\mu,S\chi))$$
$$=\alpha(0,d(\chi,S\chi),0,0,d(\mu,S\chi))$$
$$<\gamma(d(\mu,S\chi))$$
$$\therefore\ S\chi=\mu \quad (\because \text{By condition (3\&4) of lemma 3.1})$$

Since S and U weakly commute, then we get

$$d(S\mu,U\mu)=d(SU\chi,US\chi)\le d(U\chi,S\chi)=d(\mu,\mu)=0$$
$$\therefore\ S\mu=U\mu=\mu$$

Thus we have proved once again that μ is common fixed point of S, T, U and V.

If the mapping T is continuous instead of S then the proof that μ is again a common fixed point of S, T, U and V is similar.

Now let ω be a second common fixed point of S and U using inequality (2) we have,

$$d(\omega,\mu) = (d(S\omega,T\mu) \le\alpha(d(U\omega,V\mu),d(U\omega,S\omega),d(V\omega,T\mu)d(U\omega,T\mu),d(V\mu,S\omega))$$
$$= \alpha\ (d(\omega,\mu),0,0,d(\omega,\mu),d(\omega,\mu))$$
$$<\gamma(d(\omega,\mu))$$

And by condition (3 & 4) of lemma 3.1, it follows that $\omega = \mu$,

then μ is the unique common fixed point of S and U. similarly it is proved that μ is the unique common fixed point of T and V.

Hence the complete proof of the theorem.

III. CONCLUSION

In this paper, we have used weakly commuting mapping to prove some common fixed point theorems in complete metric space by taking some special case of Adrian Constantin [3], Shin-sen change [2], H.K.Pathak[8], S.Sessa[9] and R.P.Pant[1].

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